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Note on linearity of expectation

Suppose X and Y are discrete random variables over the sample space, S. $(X : S \to [0,1],\ Y : S \to [0,1])$.

Show: $\mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[X+Y]$.

Step 1: Show $\mathbb{E}[X] = \sum_{s \in S} \underbrace{X(s)}_{\text{value of } X \text{ when } s \text{ is the outcome of the experiment}} \overbrace{p(s)}^{\text{probability that } s \text{ is the outcome of the experiment}}$.

Suppose that the distinct values of X are $x_1, x_2, \ldots$

Let S_i be the event that X equals x_i. $\left(S_i = \{ s \in S : X(s) = x_i \} \right)$

set of all outcomes for which random variable X takes on the value x_i

Sum over all distinct values of X

Short hand for $\{ s \in S : X(s) = x_i \}$

$\mathbb{E}[X] = \sum_i x_i \Pr\{X = x_i\}$ — Def of expected value for discrete random variable

$= \sum_i x_i \Pr(S_i)$ — Def of S_i

$= \sum_i x_i \sum_{s \in S_i} p(s)$ — $\Pr(S_i) = \Pr\left(\bigcup_{s \in S_i} \{s\} \right) = \sum_{s \in S_i} \Pr(\{s\}) = \sum_{s \in S_i} p(s)$

disjoint events

$p(s)$ is the symbol for the probability of s being the outcome

We can express event S_i as a union of disjoint events $\{s\}$, $s \in S_i$

$= \sum_i \sum_{s \in S_i} x_i\, p(s)$ — rearrange sums

$= \sum_i \sum_{s \in S_i} X(s)\, p(s)$ — $X(s) = x_i$, the value of X for outcome s

$= \sum_{s \in S_1} X(s)\, p(s) + \sum_{s \in S_2} X(s)\, p(s) + \cdots$

Because $x_1, x_2, \ldots$ are distinct values of X,

the associated events $S_1, S_2, \ldots$ are disjoint whose union

is the entire sample space, S.

Thus, $S = \bigcup_{i=1} S_i$, where $S_i \cap S_j = \emptyset$ for any $i \neq j$.

Continuing on,

summing over all outcomes associated with event S_1

summing over all outcomes associated with event S_2, etc.

$$\mathbb{E}[X] = \sum_{s \in S_1} X(s)p(s) + \sum_{s \in S_2} X(s)p(s) + \cdots$$

$$= \sum_{s \in \{S_1 \cup S_2 \cup \ldots\}} X(s)p(s)$$

summing over all outcomes accoiated with every event

$$= \sum_{s \in S} X(s)p(s) .$$

summing over all outcomes in the sample space

<u>Ref</u>: Ross, <u>A First Course in Probability</u>, Proposition 9.1, p. 165, 8th edition.

<u>Step 2</u>: Show $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

$$\mathbb{E}[X+Y] \overset{\text{Step 1}}{=} \sum_{s \in S} \big(X(s) + Y(s)\big)\, p(s)$$

$$= \sum_{s \in S} X(s)p(s) + \sum_{s \in S} Y(s)p(s)$$

$$\overset{\text{Step 1 applied twice}}{=} \mathbb{E}[X] + \mathbb{E}[Y] .$$

$\boxtimes$

<u>Ref</u>: Ross, <u>A First Course in Probability</u>, Corollary 9.2, p. 166, 8th edition.