What are your super powerful programs/processors doing?

Logic and Proofs!  
Induction  
≡  Recursion.

What can computers do?
Work with discrete objects.  
Discrete Math  →  immense application.

Computers learn and interact with the world?
E.g. machine learning, data analysis, robotics, ... 
Probability!

My hopes and dreams.

We teach you to think more clearly and more powerfully. 
...And to deal clearly with uncertainty itself.

Probability Unit
• How can we predict unknown future events (e.g., gambling profit, next week rainfall, traffic congestion, ...)?
  – Constructive Models: Model the overall system (including the sources of uncertainty).
    • For modeling uncertainty, we’ll develop probabilistic models and techniques for analyzing them.
  – Deductive Models: Extract the “trend” from the previous outcomes (e.g., linear regression).

Wason’s experiment:1

Suppose we have four cards on a table:
▶ 1st about Alice, 2nd about Bob, 3rd Charlie, 4th Donna.
▶ Card contains person’s destination on one side, and mode of travel.
▶ Consider the theory:
  “If a person travels to Chicago, he/she flies.”
▶ Suppose you see that Alice went to Baltimore, Bob drove, Charlie went to Chicago, and Donna flew.

Which cards must you flip to test the theory?
Answer: Later.
### Propositions: Statements that are true or false.

<table>
<thead>
<tr>
<th>Proposition</th>
<th>True</th>
</tr>
</thead>
<tbody>
<tr>
<td>√2 is irrational</td>
<td>True</td>
</tr>
<tr>
<td>2 + 2 = 4</td>
<td>True</td>
</tr>
<tr>
<td>2 + 2 = 3</td>
<td>False</td>
</tr>
<tr>
<td>266th digit of pi is 4</td>
<td>False</td>
</tr>
<tr>
<td>Johnny Depp is a good actor</td>
<td>Not Proposition</td>
</tr>
<tr>
<td>Any even &gt; 2 is sum of 2 primes</td>
<td>False</td>
</tr>
<tr>
<td>x + x</td>
<td>Not a Proposition.</td>
</tr>
<tr>
<td>Alice travelled to Chicago</td>
<td>Proposition.</td>
</tr>
<tr>
<td>I love you.</td>
<td>False</td>
</tr>
</tbody>
</table>

Again: “value” of a proposition is ... True or False

### Truth Tables for Propositional Forms.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P ∧ Q</th>
<th>P ∨ Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Check: ∧ and ∨ are commutative.

One use for truth tables: Logical Equivalence of propositional forms!

Example: \((P ∨ Q)\) logically equivalent to \(¬P ∨ ¬Q\). Same Truth Table!

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>¬P</th>
<th>¬Q</th>
<th>P ∨ ¬Q</th>
<th>¬P ∨ ¬Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
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<td>T</td>
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<td>T</td>
</tr>
</tbody>
</table>

### Propositional Forms.

Put propositions together to make another...

- **Conjunction ("and")**: \(P ∧ Q\)
  - “\(P ∧ Q\)” is True when both \(P\) and \(Q\) are True. Else False.

- **Disjunction ("or")**: \(P ∨ Q\)
  - “\(P ∨ Q\)” is True when at least one \(P\) or \(Q\) is True. Else False.

- **Negation ("not")**: \(¬P\)
  - “\(¬P\)” is True when \(P\) is False. Else False.

Examples:

- \(¬(2 + 2 = 4)\) – a proposition that is ... False
- \(2 + 2 = 3 ∧ (2 + 2 = 4)\) – a proposition that is ... False
- \(2 + 2 = 3 ∨ (2 + 2 = 4)\) – a proposition that is ... True

### Quick Questions

<table>
<thead>
<tr>
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<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Is \((T ∧ Q) = Q\)? Yes? No?

Yes! Look at rows in truth table for \(P = T\).

What is \((F ∧ Q)\)? True or False.

What is \((T ∨ Q)\)? True.

What is \((F ∨ Q)\)? True.

### Put them together..

**Propositions:**

- \(P_1\) - Person 1 rides the bus.
- \(P_2\) - Person 2 rides the bus.

---

But we can’t have either of the following happen: That either person 1 or person 2 ride the bus and person 3 or 4 ride the bus. Or that person 2 or person 3 ride the bus and that either person 4 rides the bus or person 5 doesn’t.

**Propositional Form:**

\(¬((P_1 ∨ P_2) ∧ (P_3 ∨ P_4)) ∨ ((P_2 ∨ P_3) ∧ (P_4 ∨ ¬P_5)))\)

Can person 3 ride the bus?

Can person 3 and person 4 ride the bus together?

This seems ... complicated.

We can program!!!

We need a way to keep track!

### Distributive?

\(P ∧ (Q ∨ R) = (P ∧ Q) ∨ (P ∧ R)\)

Simplify: \((T ∧ Q) = Q, (F ∧ Q) = F\).

**Cases:**

- \(P\) is True .
  - LHS: \(T ∧ (Q ∨ R) = (Q ∨ R)\)
  - RHS: \((T ∧ Q) ∨ (T ∧ R) = (Q ∨ R)\).
- \(P\) is False .
  - LHS: \(F ∧ (Q ∨ R) = F\)
  - RHS: \((F ∧ Q) ∨ (F ∧ R) = (F ∨ F) = F\).

\(P ∨ (Q ∨ R) = (P ∨ Q) ∨ (P ∨ R)\)

Simplify: \(T ∨ Q = T, F ∨ Q = Q\). ...

**Foil 1:**

\((A ∨ B) ∧ (C ∨ D) = (A ∧ C) ∨ (A ∧ D) ∨ (B ∧ C) ∨ (B ∧ D)\)

**Foil 2:**

\((A ∨ B) ∨ (C ∧ D) = (A ∨ C) ∨ (A ∨ D) ∨ (B ∧ C) ∨ (B ∧ D)\)
Implication.

\[ P \implies Q \text{ interpreted as } \]

- If \( P \), then \( Q \).
- True Statements: \( P, P \implies Q \).
- Conclude: \( Q \) is true.

Examples:

- Statement: If you stand in the rain, then you'll get wet.
  - \( P = \) "you stand in the rain"
  - \( Q = \) "you will get wet"
- Statement: "Stand in the rain"
  - Can conclude: "you'll get wet."

Statement:

- If a right triangle has sidelengths \( a \leq b \leq c \), then \( a^2 + b^2 = c^2 \).
- \( P = \) "a right triangle has sidelengths \( a \leq b \leq c \),
  - \( Q = \) "\( a^2 + b^2 = c^2 \)."

Non-Consequences/consequences of Implication

The statement "\( P \implies Q \)"

- only is False if \( P \) is True and \( Q \) is False .
- False implies nothing
- \( P \) False means \( Q \) can be True or False
- Anything implies true.
- \( P \) can be True or False when \( Q \) is True

If chemical plant pollutes river, fish die.
If fish die, did chemical plant pollute river?
Not necessarily.

- \( P \implies Q \) and \( Q \) are True does not mean \( P \) is True
  - Be careful!
- Instead we have:
  - \( P \implies Q \) and \( Q \) are True does mean \( P \) is True.

The chemical plant pollutes river. Can we conclude fish die?
Some Fun: use propositional formulas to describe implication?

\( ((P \implies Q) \land P) \implies Q \).

Contrapositive, Converse

- Contrapositive of \( P \implies Q \) is \( \neg Q \implies \neg P \).
- If the plant pollutes, fish die.
- If the fish don’t die, the plant does not pollute.
  - (contrapositive)
- If you stand in the rain, you get wet.
- If you did not get wet, you did not stand in the rain.
  - (not contrapositive!)
- If \( P \) is false, then we know that \( Q \) is false.

- Converse of \( P \implies Q \) is \( Q \implies P \).
If fish die the plant pollutes.
Not logically equivalent!

- Definition: If \( P \implies Q \) and \( Q \implies P \) is \( P \) if and only if \( Q \) or \( P \implies Q \).
  - (Logically Equivalent: \( \iff \) )

Implication and English.

\( P \implies Q \)

Poll.

- If \( P \), then \( Q \).
- \( Q \) if \( P \).
- Just reversing the order.
- \( P \) only if \( Q \).
- Remember if \( P \) is true then \( Q \) must be true.
  - this suggests that \( P \) can only be true if \( Q \) is true.
  - since if \( Q \) is false \( P \) must have been false.
- \( P \) is sufficient for \( Q \).
  - This means that proving \( P \) allows you to conclude that \( Q \) is true.
  - Example: Showing \( n > 4 \) is sufficient for showing \( n > 3 \).
- \( Q \) is necessary for \( P \).
  - For \( P \) to be true it is necessary that \( Q \) is true.
  - Or if \( Q \) is false then we know that \( P \) is false.
  - Example: It is necessary that \( n > 3 \) for \( n > 4 \).

Variables.

Prepositions?

- \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)
- \( x > 2 \)
- \( n \) is even and the sum of two primes

No. They have a free variable.

We call them predicates, e.g., \( Q(x) = "x \text{ is even}" \)

Same as boolean valued functions from 61A!

- \( P(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)
- \( R(x) = "x > 2" \)
- \( Q(n) = "n \text{ is even and the sum of two primes}" \)

Remember Wason’s experiment!

- \( F(x) = "x \text{ flew}" \)
- \( C(x) = "x \text{ went to Chicago}" \)

Next: Statements about boolean valued functions!!
Quantifiers... 

There exists quantifier:
\((\exists x \in S)(P(x))\) means “There exists an \(x\) in \(S\) where \(P(x)\) is true.”

For example:
\((\exists x \in N)(x = x^2)\)
Equivalent to “\((0 = 0) \lor (1 = 1) \lor (2 = 4) \lor \ldots\)"

Much shorter to use a quantifier!

For all quantifier:
\((\forall x \in S)(P(x))\), means “For all \(x\) in \(S\), \(P(x)\) is true.”

Examples:
“Adding 1 makes a bigger number.”
\((\forall x \in N)(x + 1 > x)\)
“The square of a number is always non-negative”
\((\forall x \in N)(x^2 \geq 0)\)

Wait! What is \(N\)?

More for all quantifiers examples.

- “Doubling a number always makes it larger”
  \((\forall x \in N)(2x > x)\) \text{ False} \quad \text{Consider } x = 0

  Can fix statement...
  \((\forall x \in N)(2x \geq x)\) \text{ True}

- “Square of any natural number greater than 5 is greater than 25.”
  \((\forall x \in N)(x > 5 \implies x^2 > 25)\)

  Idea alert: Restrict domain using implication.
  Later we may omit universe if clear from context.

Quantifiers: universes.

Proposition: “For all natural numbers \(n\), \(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\)”

Proposition has universe: “the natural numbers”.

Universe examples include...
- \(N = \{0, 1, \ldots\}\) (natural numbers).
- \(Z = \{\ldots, -1, 0, 1, \ldots\}\) (integers)
- \(Z^+\) (positive integers)
- \(R\) (real numbers)
- Any set: \(S = \{\text{Alice, Bob, Charlie, Donna}\}\).

See note 0 for more!

Other proposition notation (for discussion):
- “\(d|n\)” means \(d\) divides \(n\)
  or \(\exists k \in Z, n = kd\).
- \(2|4\) True.
- \(4|2\) False.

Quantifiers...not commutative.

- In English: “there is a natural number that is the square of every natural number”.
  \((\exists y \in N)(\forall x \in N)(y = x^2)\) \text{ False}

  Can fix statement...
  \((\forall x \in N)(\exists y \in N)(y = x^2)\) \text{ False}

- In English: “the square of every natural number is a natural number”.
  \((\forall x \in N)(\exists y \in N)(y = x^2)\) \text{ True}

Back to: Wason’s experiment:1

Theory: “If a person travels to Chicago, he/she flies.”

Alice to Baltimore, Bob drove, Charlie to Chicago, and Donna flew.
Which cards do you need to flip to test the theory?

- Chicago(D) → Flew(D) is true when Flew(D) is true.
- Alice to Baltimore, Bob drove, Charlie to Chicago, and Donna flew.

Consider \((\forall x \in S)(P(x))\).

English: there is an \(x\) in \(S\) where \(P(x)\) does not hold.
That is,
\(-\forall x \in S)(P(x))\) \iff \(\exists x \in S)(\neg P(x))\).

What we do in this course! We consider claims.

Claim: \((\forall x) P(x)\) “For all inputs \(x\) the program works.”
For False, find \(x\), where \(\neg P(x)\).

Counterexample.
Bad input.
Case that illustrates bug.
For True: prove claim. Next lectures...

Quantifiers.....notion...DeMorgan again.

Consider
\(-\forall x \in S)(P(x))\).

English: there is an \(x\) in \(S\) where \(P(x)\) does not hold.

That is,
\(-\forall x \in S)(P(x))\) \iff \(\exists x \in S)(\neg P(x))\).

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Claim: \((\forall x) P(x)\) “For all inputs \(x\) the program works.”
For False, find \(x\), where \(\neg P(x)\).

Counterexample.
Bad input.
Case that illustrates bug.
For True: prove claim. Next lectures...
Negation of exists.

Consider
\[ \neg(\exists x \in S)(P(x)) \]
English: means that there is no \( x \in S \) where \( P(x) \) is true.
That is,
\[ \neg(\exists x \in S)(P(x)) \iff \forall (x \in S) \neg P(x). \]

Which Theorem?

Theorem: \((\forall n \in \mathbb{N}) \neg(\exists a, b, c \in \mathbb{N}) (n \geq 3 \implies a^n + b^n = c^n)\)
Which Theorem?
Fermat's Last Theorem!
Remember Special Triangles: for \( n = 2 \), we have 3,4,5 and 5,7, 12 and ...
1637: Proof doesn’t fit in the margins.
1993: Wiles ...(based in part on Ribet’s Theorem)
DeMorgan Restatement:
Theorem: \((\neg (\exists n \in \mathbb{N}) (\exists a, b, c \in \mathbb{N}) (n \geq 3 \implies a^n + b^n = c^n)\)

Summary.
Propositions are statements that are true or false.
Propositional forms use \( \land, \lor, \neg \).
Propositional forms correspond to truth tables.
Logical equivalence of forms means same truth tables.
Implication: \( P \implies Q \iff \neg P \lor Q \).
Contrapositive: \( \neg Q \implies \neg P \)
Converse: \( Q \implies P \)
Predicates: Statements with “free” variables.
Quantifiers: \( \forall x \ P(x) , \exists y \ Q(y) \)
Now can state theorems! And disprove false ones!
DeMorgan Laws: “Flip and Distribute negation”
\[ \neg( P \lor Q ) \iff (\neg P \land \neg Q) \]
\[ \neg(\exists x P(x)) \iff \forall x \neg P(x). \]
Next Time: proofs!