Today.

Polynomials.

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Two points make a line.

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The idea of the day.

Two points make a line.
Lots of lines go through one point.

Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

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Polynomials $P(x)$ with arithmetic modulo p: [1] $a_i \in \{0, \dots, p-1\}$ and

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod{p},$$

for $x \in \{0, \dots, p-1\}$.

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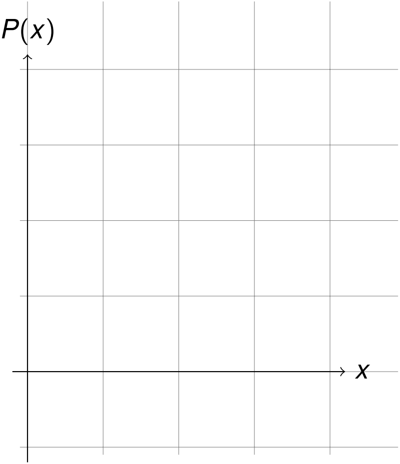

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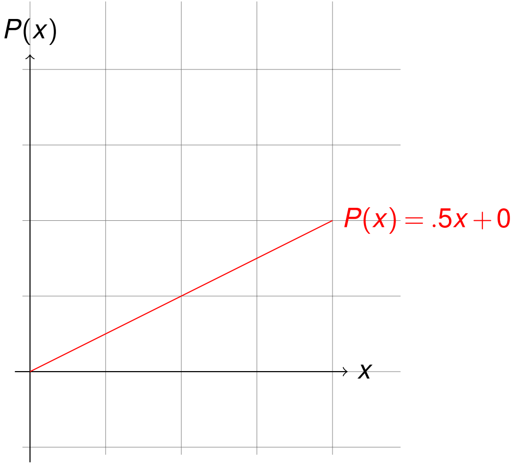

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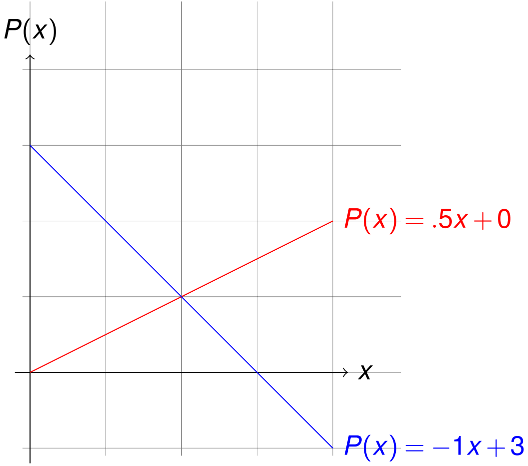

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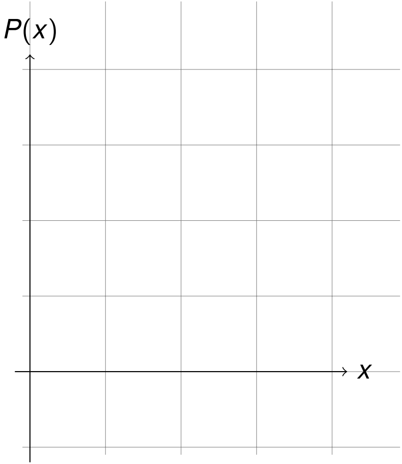

Parabola: $P(x) = a_2 x^2 + a_1 x + a_0$

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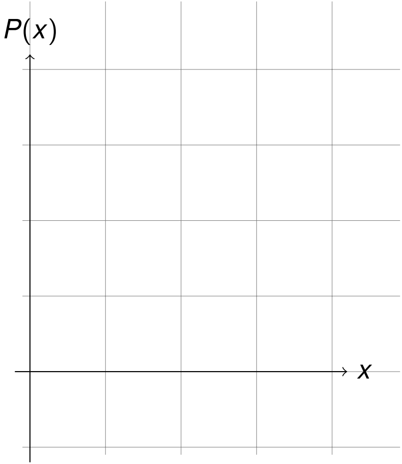

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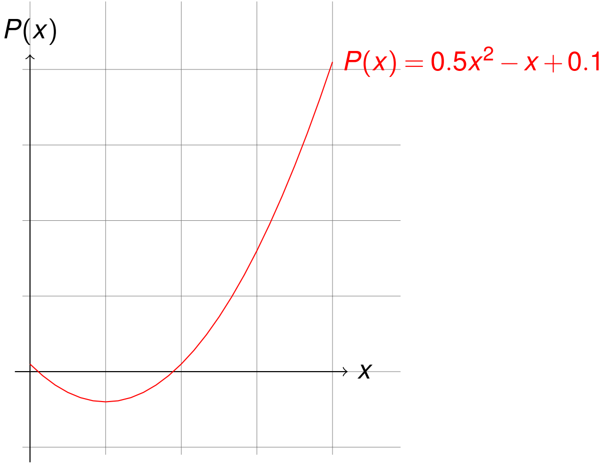

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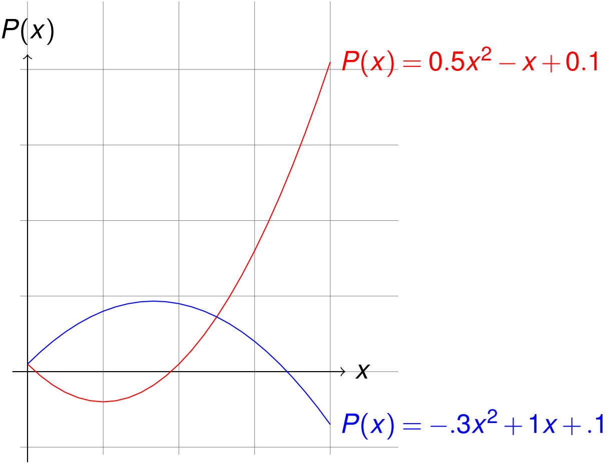

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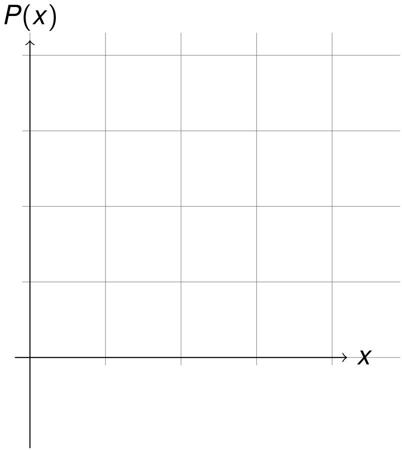

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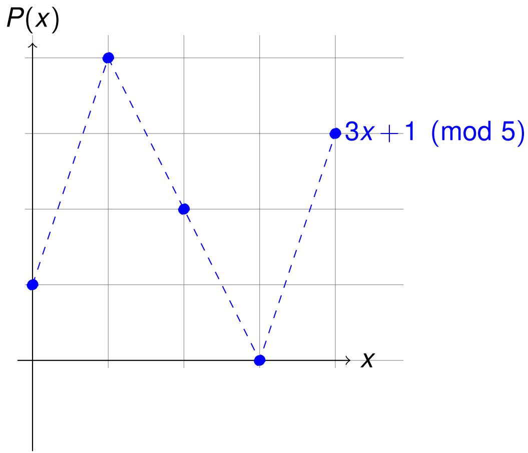

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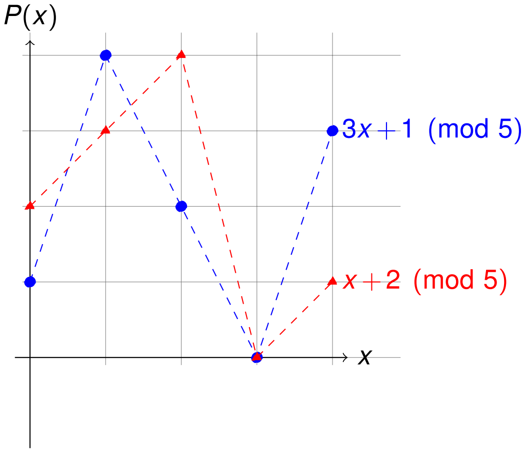

Finding an intersection.

$x + 2 \equiv 3x + 1 \pmod{5}$

$\implies 2x \equiv 1 \pmod{5}$

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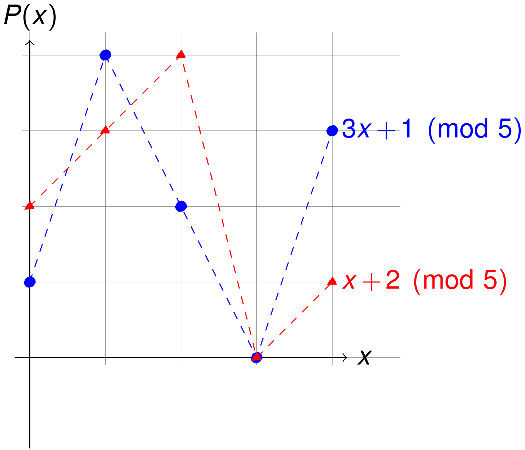

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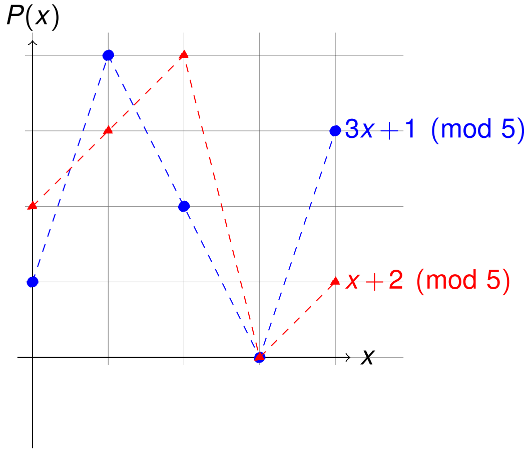

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Good when modulus is prime!!

Two points make a line.

Fact: Exactly 1 degree $\leq d$ polynomial contains $d+1$ points. [2]

[2] Points with different x values.

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Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

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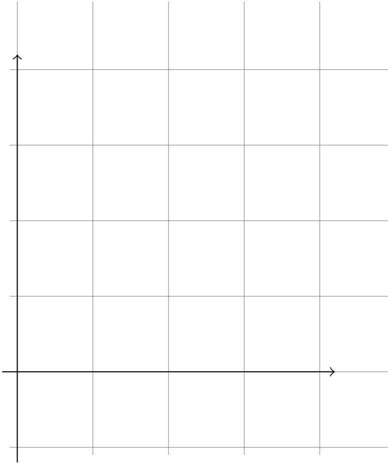
3 points determine a parabola.



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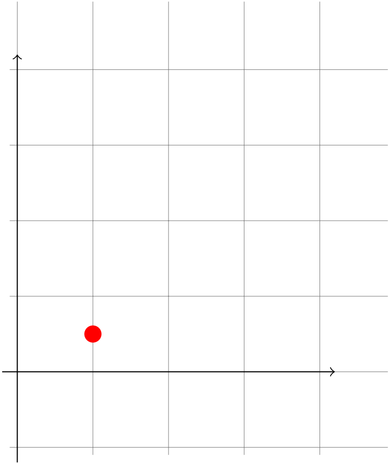

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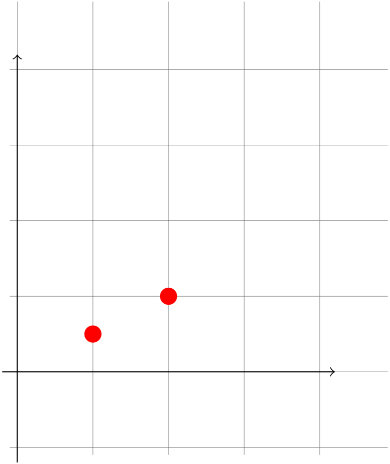

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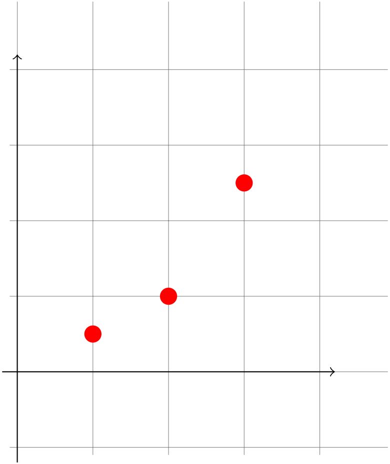

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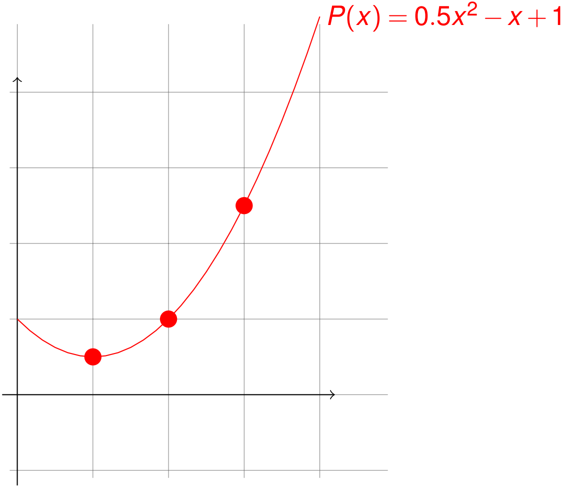

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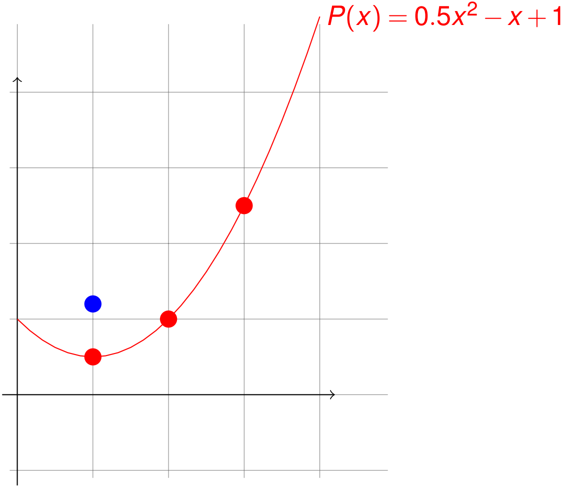

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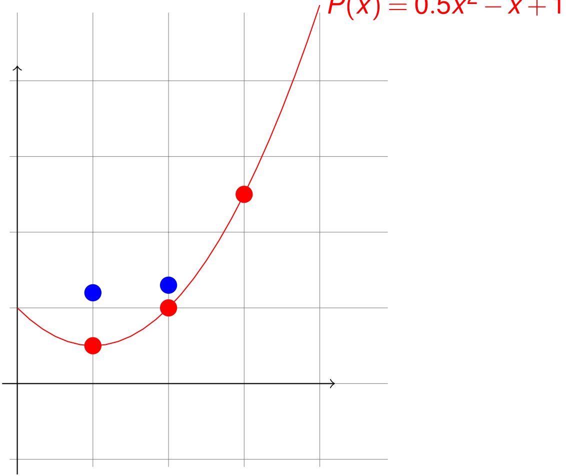

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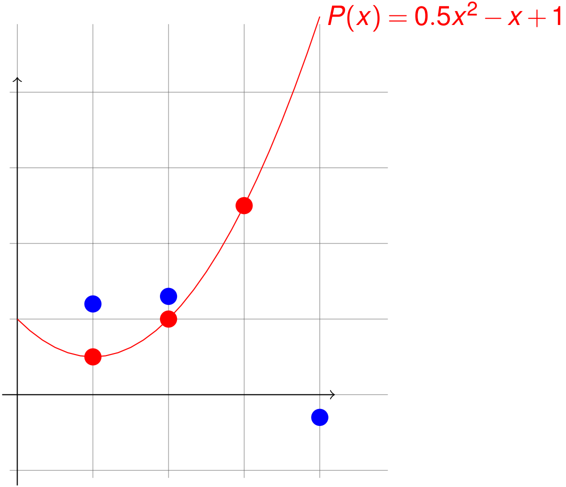

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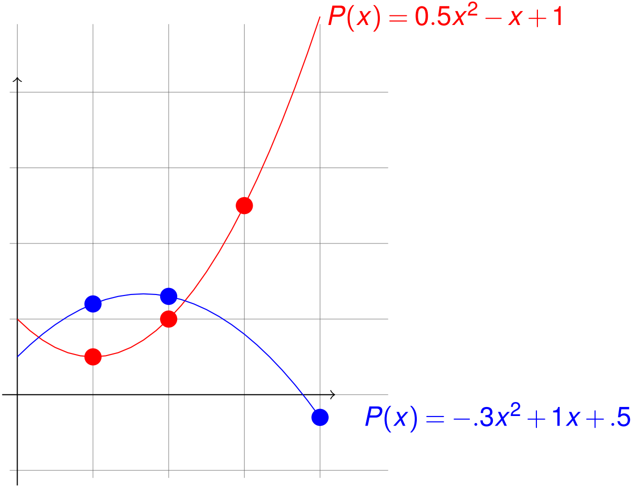

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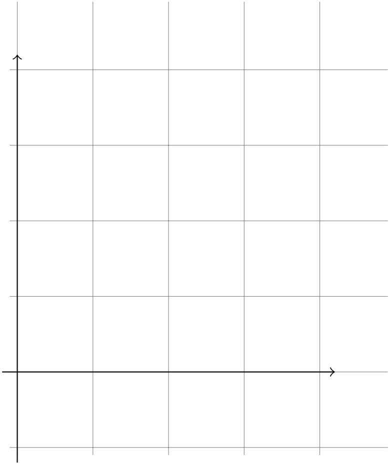

There is $P(x)$ contains blue points and *any* $(0, y)$!

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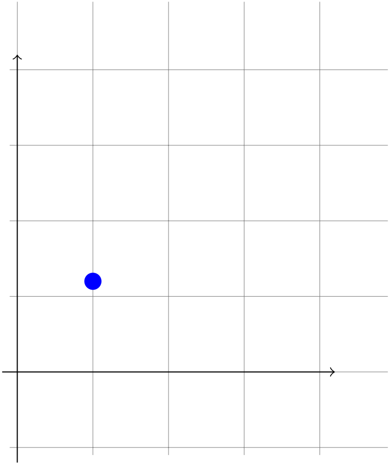

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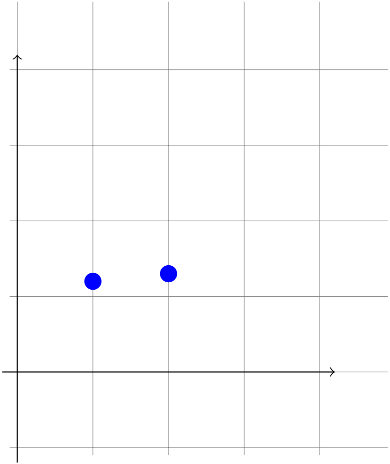

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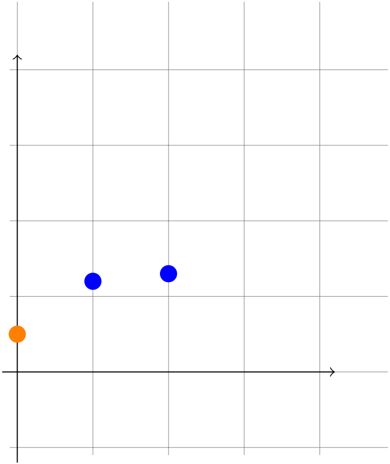

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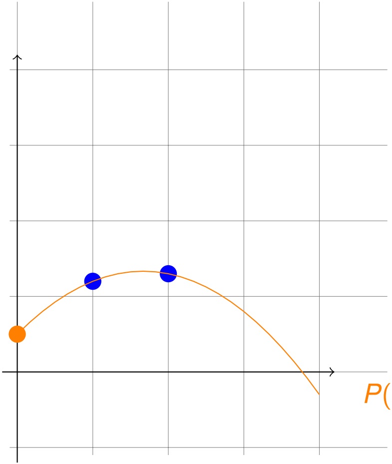

$P(x) = -.3x^2 + 1x + .5$

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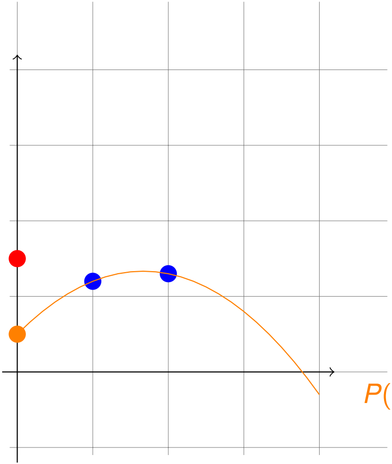

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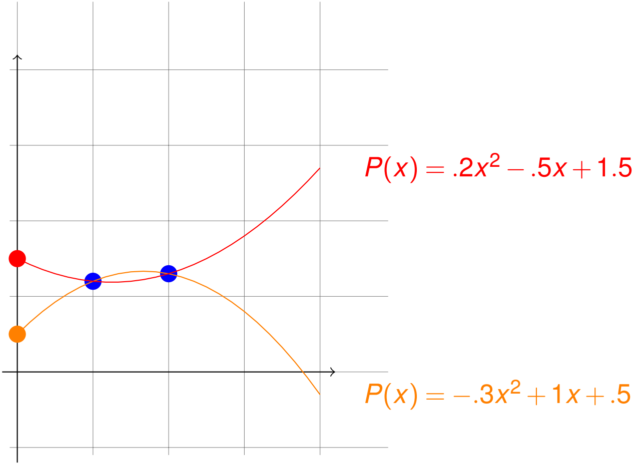

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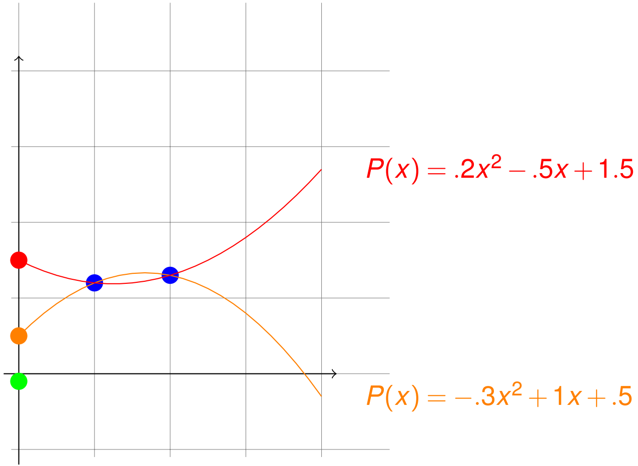

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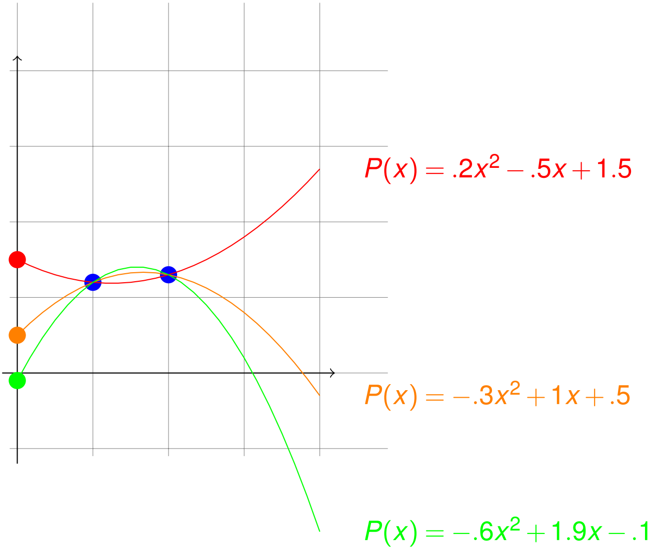

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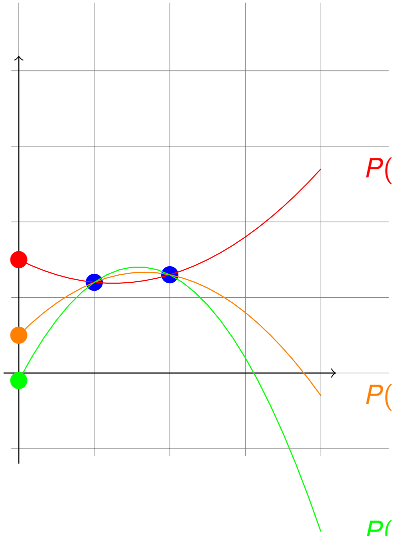

$P(x) = .2x^2 - .5x + 1.5$

$P(x) = -.3x^2 + 1x + .5$

$P(x) = -.6x^2 + 1.9x - .1$

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Modular Arithmetic Fact and Secrets

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Roubustness: Any k shares gives secret.
Knowing k pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.

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Secret $s \in \{0,\ldots,p-1\}$

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We will work with polynomials with arithmetic modulo p.

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For set of x-values, $x_1, \ldots, x_{d+1}$.

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Construction proves the existence of a polynomial!

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$$= 2(x-3) = 2x - 6 = 2x + 4 \pmod 5.$$

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3);(2,4);(3,0)$.

Work modulo 5.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

$$\Delta_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2} = 3(x-2)(x-3)$$
$$= 3x^2 + 3 \pmod 5$$

Put the delta functions together.

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Construction proves the existence of the polynomial!

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Uniqueness Fact. At most one degree d polynomial hits $d+1$ points.

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Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

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That is, $P(x) = (x - a)Q(x) + r$

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Proof works for reals, rationals, and complex numbers.

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Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

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Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

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(Almost) the same as what is missing: one $P(i)$.

Runtime.

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Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $k-1$ polynomial n times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of k equations using $\log p$-bit arithmetic.

A bit more counting.

What is the number of degree d polynomials over $GF(m)$?

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Erasure Codes.

Satellite

GPS device

Erasure Codes.

Satellite

3 packet message.

GPS device

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3 packet message.

Lose 3 out 6 packets.

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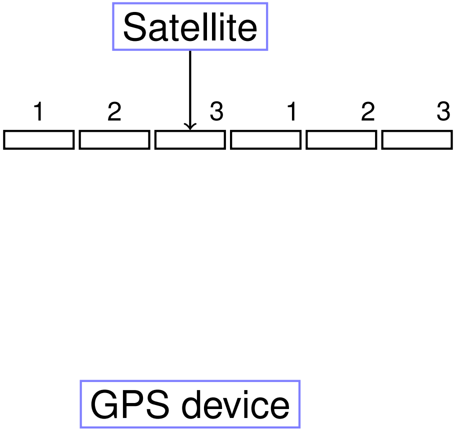
Erasure Codes.



3 packet message. So send 6!

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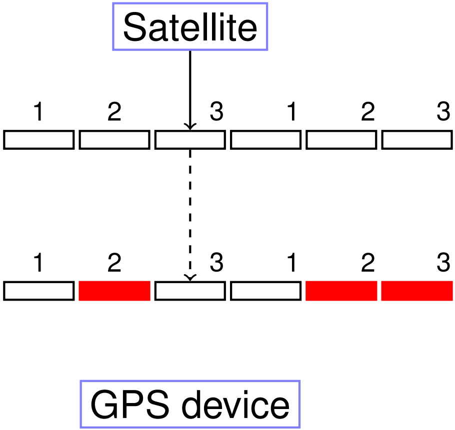

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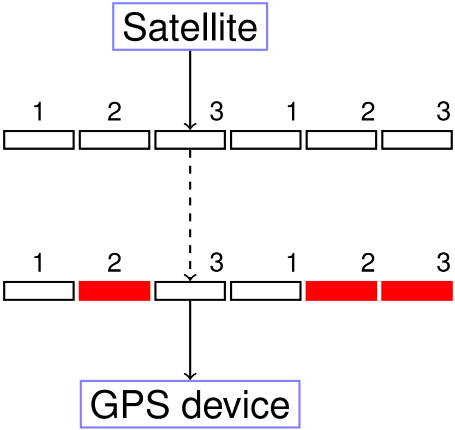

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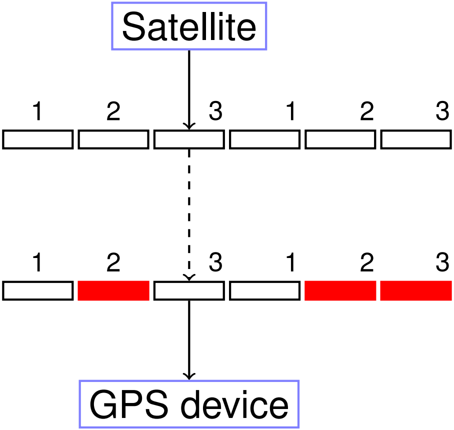

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Gets packets 1,1,and 3.

Solution Idea.

n packet message, channel that loses k packets.

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Must send $n+k$ packets!

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- Any n packets should allow reconstruction of n packet message.
- Any n point values allow reconstruction of degree $n-1$ polynomial.

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Alright!

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- Any n packets should allow reconstruction of n packet message.
- Any n point values allow reconstruction of degree $n-1$ polynomial.

Alright!!

Solution Idea.

n packet message, channel that loses k packets.

Must send $n+k$ packets!

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Use polynomials.

Problem: Want to send a message with n packets.

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Erasure Coding Scheme: message = $m_0, m_2, \ldots, m_{n-1}$.

1. Choose prime $p \approx 2^b$ for packet size b.
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Erasure Codes.

Satellite

GPS device

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Satellite

n packet message.

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Lose k packets.

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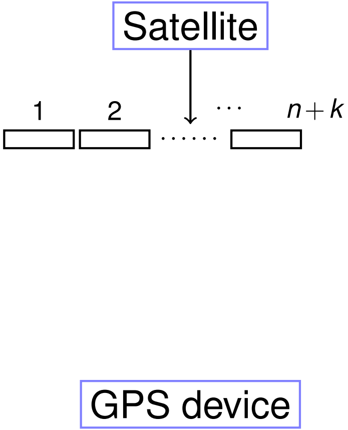
Erasure Codes.



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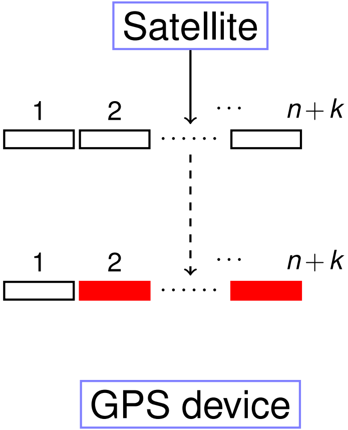

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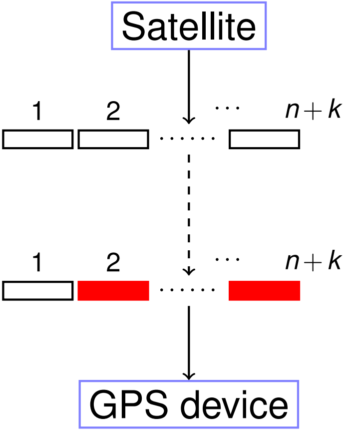

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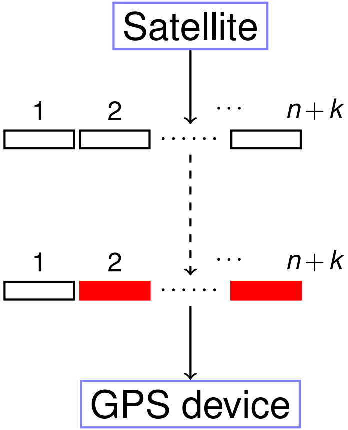

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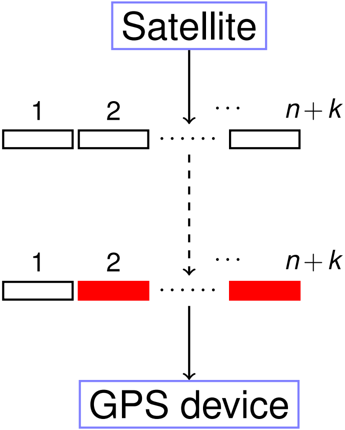

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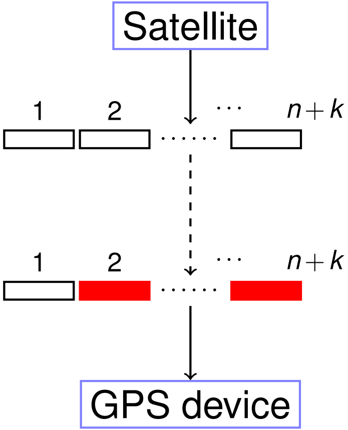

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$a_1 = 2a_0$. $a_0 = 2 \pmod 7$ $a_1 = 4 \pmod 7$ $a_2 = 2 \pmod 7$

$P(x) = 2x^2 + 4x + 2$

$P(1) = 1$, $P(2) = 4$, and $P(3) = 4$

Send
Packets: $(1,1),(2,4),(3,4),(4,7),(5,2),(6,0)$

Example

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

Modulo 7 to accommodate at least 6 packets.

Linear equations:

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Notice that packets contain “x-values”.

Bad reception!

Send: $(1,1),(2,4),(3,4),(4,7),(5,2),(6,0)$

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Reconstruct?

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Format: $(i, R(i))$.

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Lagrange or linear equations.

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Channeling Sahai

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Channeling Sahai ...

$P(x) = 2x^2 + 4x + 2$

Message? $P(1) = 1,$

Bad reception!

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Questions for Review

You want to encode a secret consisting of 1,4,4.

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Send n packets b-bit packets, with k errors.

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How big should modulus be?
Larger than 8 and prime!

Send n packets b-bit packets, with k errors.
Modulus should be larger than $n+k$ and also larger than 2^b.

Polynomials.

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- ..give Secret Sharing.

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Noisy Channel: corrupts k packets. (rather than loss.)

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Additional Challenge: Finding which packets are corrupt.

Error Correction

Satellite

GPS device

Error Correction

Satellite

3 packet message.

GPS device

Error Correction

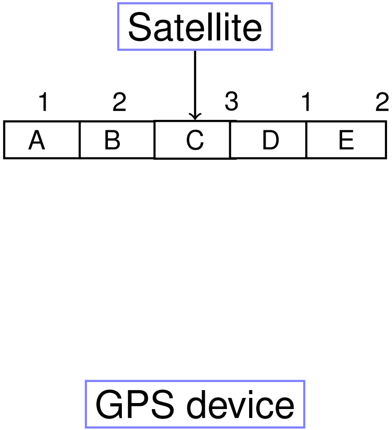
Satellite

3 packet message.

Corrupts 1 packets.

GPS device

Error Correction



3 packet message. Send 5.

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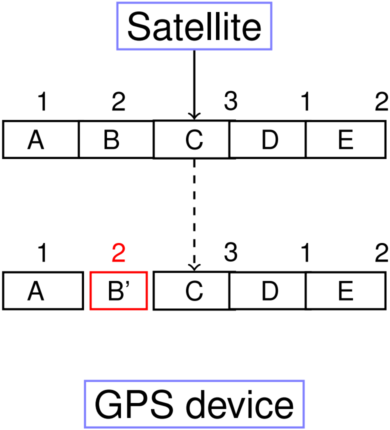
Error Correction



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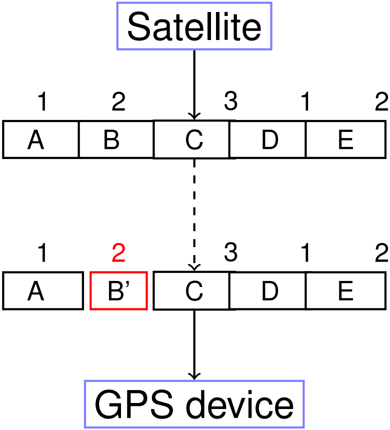

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Properties: proof.

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Total points contained by both: $2n+2k$.

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Total points to choose from : $n+2k$.

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Total points to choose from : $n+2k$. H Holes.

Properties: proof.

$P(x)$: degree $n-1$ polynomial.
Send $P(1),\ldots,P(n+2k)$
Receive $R(1),\ldots,R(n+2k)$
At most k i's where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n+k$ points i,
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that contains $\geq n+k$ received points.

Proof:
(1) Sure. Only k corruptions.
(2) Degree $n-1$ polynomial $Q(x)$ consistent with $n+k$ points.
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Receive $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$.

$P(i) = R(i)$ for $n + k = 3 + 1 = 4$ points.

Slow solution.

Brute Force:
For each subset of $n+k$ points

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Reconstructs $P(x)$ and only $P(x)$!!

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Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

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All equations..

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p_2 + p_1 + p_0 &\equiv 3 \pmod 7 \\
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Something like $(n/k)^k$ ...Exponential in k!.

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How do we find where the bad packets are efficiently?!?!?!

Ditty...

Ditty...

Where oh where

Ditty...

Where oh where can my bad packets be ...

Ditty...

Where oh where can my bad packets be ...

Ditty...

Where oh where can my bad packets be ...

On Tuesday.