The future in this course.

What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide. Chances?

(A) Red Probability is 3/8
(B) Blue probability is 3/9
(C) Yellow Probability is 2/8
(D) Blue probability is 3/8

Today: Counting!

Outline: basics

1. Counting.
2. Tree
3. Rules of Counting
4. Sample with/without replacement where order does/doesn’t matter.

Probability is soon... but first let’s count.

Count?

How many outcomes possible for k coin tosses?
How many poker hands?
How many handshakes for n people?
How many diagonals in a n sided convex polygon?
How many 10 digit numbers?
How many 10 digit numbers without repetition?
How many ways can I divide up 5 dollars among 3 people?

Using a tree...

How many 3-bit strings?
   How many different sequences of three bits from {0, 1}? 
   How would you make one sequence?
   How many different ways to do that making?

8 leaves which is 2 × 2 × 2. One leaf for each string.
8 3-bit strings!

First Rule of Counting: Product Rule

Objects made by choosing from n₁, then n₂, ..., then nₖ 
the number of objects is n₁ × n₂ × ··· × nₖ.

In picture, 2 × 2 × 3 = 12!
Poll

Mark what correct.
(A) \(|10\text{ digit numbers}| = 10^{10}\)
(B) \(|k\text{ coin tosses}| = 2^k\)
(C) \(|10\text{ digit numbers}| = 9 \cdot 10^9\)
(D) \(|n\text{ digit base }m\text{ numbers}| = m^n\)
(E) \(|n\text{ digit base }m\text{ numbers}| = (m-1)m^n

(A) or (C)? (D) or (E)? (C) are correct.

Permutations.

How many 10 digit numbers without repeating a digit?
10 ways for first, 9 ways for second, 8 ways for third, ...
\[ \ldots 10 \cdot 9 \cdot 8 \cdot \ldots 1 = 10! \]  
How many different samples of size \(k\) from \(n\) numbers without replacement.
\(n\) ways for first choice, \(n-1\) ways for second, \(n-2\) choices for third, ...
\[ n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = \frac{n!}{(n-k)!} \]
How many orderings of \(n\) objects are there?
Permutations of \(n\) objects.
\(n\) ways for first, \(n-1\) ways for second, \(n-2\) ways for third, ...
\[ n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n! \]

One-to-One Functions.

How many one-to-one functions from \(|S|\) to \(|S|\).
\(|S|\) choices for \(f(s_1)\), \(|S| - 1\) choices for \(f(s_2)\), ...
So total number is \(|S| \times |S| - 1 \cdot 1 = |S|!\)
A one-to-one function is a permutation!

Using the first rule..

How many outcomes possible for \(k\) coin tosses?
2 ways for first choice, 2 ways for second choice,...
\[ 2 \times 2 \times \ldots 2 = 2^k \]
How many 10 digit numbers?
10 ways for first choice, 10 ways for second choice,...
\[ 10 \times 10 \times \ldots 10 = 10^k \]
How many \(n\) digit base \(m\) numbers?
\(m\) ways for first, \(m\) ways for second,...
\[ m^n \]
(Is 09, a two digit number?)
If no. Then \((m-1)m^{n-1}\).

Functions, polynomials.

How many functions \(f\) mapping \(|S|\) to \(|T|\)?
\(|T|\) ways to choose for \(f(s_1)\), \(|T|\) ways to choose for \(f(s_2)\), ...
\[ |T|^{\text{rd}} \]
How many polynomials of degree \(d\) modulo \(p\)?
\(p\) ways to choose for first coefficient, \(p\) ways for second, ...
\[ p^{d+1} \]
\(p\) values for first point, \(p\) values for second, ...
\[ p^{d+1} \]
Questions?

Counting sets..when order doesn't matter.

How many poker hands?
\[ 52 \times 51 \times 50 \times 49 \times 48 \]  
Are A, K, Q, 10, J of spades
and 10, J, O, K, A of spades the same?
Second Rule of Counting: If order doesn't matter count ordered objects and then divide by number of orderings.  
Number of orderings for a poker hand: "5!"
(The "!" means factorial, not Exclamation.)
\[ \frac{52 \times 51 \times 50 \times 49 \times 48}{5!} \]
Can write as...
\[ \frac{52!}{5!} \]
\[ \frac{51 \times 4!}{5!} \]
Generic: ways to choose 5 out of 52 possibilities.

\[ ^{1}\text{By definition: } 0! = 1. \]
Ordered to unordered.

Second Rule of Counting: If order doesn’t matter count ordered objects and then divide by number of orderings.

- How many red nodes (ordered objects)? 9.
- How many red nodes mapped to one blue node? 3.
- How many blue nodes (unordered objects)? \( \frac{9}{3} = 3 \).
- How many poker deals? 52 \( \times \) 51 \( \times \) 50 \( \times \) 49 \( \times \) 48.
- How many poker deals per hand? Map each deal to ordered deal: 5!
- How many poker hands? \( \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = \frac{52 \times 51 \times 50 \times 49 \times 48}{120} \).
- Questions?

Example: Anagram

First rule: \( n_1 \times n_2 \times \cdots \times n_p \), Product Rule.
Second rule: when order doesn’t matter divide...

- A’s are the same!
- What is \( \Delta \)? ANAGRAM
  \[ A _ 1 N A _ 2 G R A _ 3 M , A _ 2 N A _ 1 G R A _ 3 M , \ldots \]
  \( \Delta = 3 \times 2 \times 1 = 3! \) First rule!
⇒ 7! Second rule!

秩序のない一組のハンドへの対応

元の順列の数: \( n_1 \times n_2 \times \cdots \times n_p \)
第2法則: 場合の順序が問題なければ、順列の数を除く

- 第1法則: \( n_1 \times n_2 \times \cdots \times n_p \)
- 第2法則: 場合の順序が問題なければ、順列の数を除く

Some Practice.

- How many orderings of letters of CAT?
  3 ways to choose first letter, 2 ways for second, 1 for last.
  ⇒ \( 3 \times 2 \times 1 = 3! \) orderings
- How many orderings of the letters in ANAGRAM?
  Ordered, except for A!
  \( \frac{7!}{2!} \) total orderings of 7 letters. 7!
  Total “extra counts” or orderings of three A’s? 3!
  Total orderings? \( \frac{7!}{2!} \)
- How many orderings of MISSISSIPPI?
  4 S’s, 4 I’s, 2 P’s.
  11 letters total.
  \( \frac{11!}{4! \times 4! \times 2!} \) ordered objects per “unordered object”
  ⇒ \( \frac{11!}{4! \times 4! \times 2!} \)
Sampling...

Sample k items out of n
Without replacement:
Order matters: \( n \times n-1 \times n-2 \times \ldots \times n-k+1 = \frac{n!}{(n-k)!} \)
Order does not matter:
Second Rule: divide by number of orders – “k!”
\[ \frac{n!}{k!(n-k)!} \]

With Replacement.
Order matters: \( n \times n \times \ldots \times n = n^k \)
Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!
Different number of unordered elts map to each unordered elt.

Unordered elt: 1, 2, 3
3! ordered elts map to it.
Unordered elt: 1, 2, 2
2! ordered elts map to it.

How do we deal with this mess??

Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice(2^5), divide out order ???
5 dollars for Bob and 0 for Alice:
one ordered set: (B, B, B, B, B).
4 for Bob and 1 for Alice:
5 ordered sets: (A, B, B, B, B); (B, A, B, B, B); ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.
(A, B, B, B, B): 5; (A, B, B, B, B): (B, A, B, B, B);...
(A, A, B, B, B): \( \binom{3}{2} \); (A, A, B, B, B); (A, B, A, B, B); (A, B, B, A, B);... and so on.

Second rule of counting is no good here!

Stars and Bars.

How many different 5 star and 2 bar diagrams?
| ⋆ | ⋆ ⋆ ⋆ ⋆ .
7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4
| ⋆ | ⋆ ⋆ ⋆ . Bars in first and third position.
Alice: 1; Bob 4; Eve: 0
⋆ | ⋆ ⋆ ⋆ . Bars in second and seventh position.

\( \binom{7}{2} \) ways to do so and
\( \binom{5}{5} \) ways to split 5 dollars among 3 people.

How do we add up n numbers to sum to k? or
“k from n with replacement where order doesn’t matter”
In general, k stars n-1 bars.

\[ \binom{n+k-1}{k} \]

Or: k unordered choices from set of n possibilities with replacement.
Sample with replacement where order doesn’t matter.

Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: ...
Stars and Bars: * * * * *
Each split “is” a sequence of stars and bars.
Each sequence of stars and bars “is” a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

Summary.

First rule: \( n_1 \times n_2 \times \cdots \times n_k \).
k Samples with replacement from n items: \( n^k \).
Sample without replacement: \( \binom{n+k-1}{k} \).

Second rule: when order doesn’t matter (sometimes) can divide...
Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

“One-to-one rule: equal in number if one-to-one correspondence.
pause Bijection!
Sample k times from n objects with replacement and order doesn’t matter: \( \binom{n+k-1}{k} \).
Poll

Mark what's correct.

(A) ways to split $k$ dollars among $n$: \( \binom{k+n-1}{n-1} \)
(B) ways to split $n$ dollars among $k$: \( \binom{n+k-1}{k-1} \)
(C) ways to split 5 dollars among 3: \( \binom{5+3-1}{3-1} \)
(D) ways to split 5 dollars among 3: \( \binom{7}{5} \)

All correct.

Quick review of the basics.

First rule: \( n_1 \times n_2 \times \cdots \times n_3 \).

$k$ samples with replacement from $n$ items: \( n^k \).

Sample without replacement: \( \frac{n!}{(n-k)!} \).

Second rule: when order doesn't matter divide..when possible.

Sample without replacement and order doesn't matter: \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

"$n$ choose $k$"

One-to-one rule: equal in number if one-to-one correspondence.

Sample with replacement and order doesn't matter: \( \binom{k+n-1}{n-1} \).

Distribute $k$ samples (stars) over $n$ possibilities ($n-1$ bars group possibilities.)

Distribute $k$ dollars to $n$ people.