Today.

Goodbye Modular Arithmetic!

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Countability and Uncountability.

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Countability and Uncountability.

Computability.

Summary: polynomials.

Set of $d+1$ points determines degree d polynomial.

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Encode secret using degree $k-1$ polynomial:

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Other Algebraic-Geometric codes.

Farewell (for now) to modular arithmetic...

Modular arithmetic modulo a prime.

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For modular arithmetic...no Calculus. Sad face!

Next up: how big is infinity.

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- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

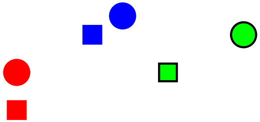
Infinite!

How big are the reals or the integers?

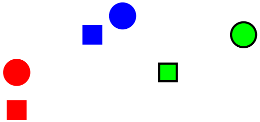
Infinite!

Is one bigger or smaller?

Same size?



Same size?



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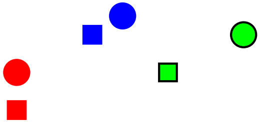
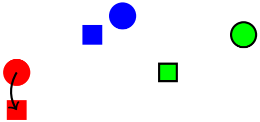


Same number?

Make a function f : Circles $\to$ Squares.

Same size?

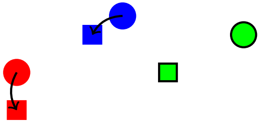


Same number?

Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

Same size?



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Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

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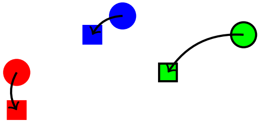


Same number?

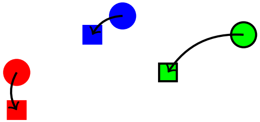
Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

Same size?



Same number?

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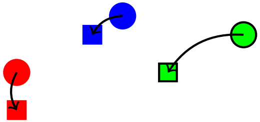
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one.

Same size?



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Make a function f : Circles $\rightarrow$ Squares.

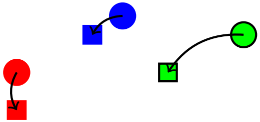
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one. Each circle mapped to different square.

Same size?



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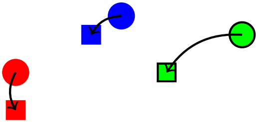
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

Same size?



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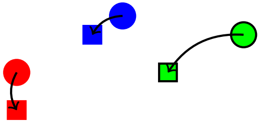

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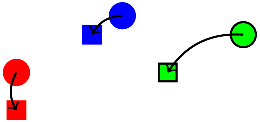
Onto. Each square mapped to from some circle .

Same size?



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Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

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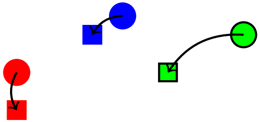

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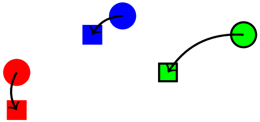

Same number?
Make a function f : Circles $\rightarrow$ Squares.
f(red circle) $=$ red square
f(blue circle) $=$ blue square
f(circle with black border) $=$ square with black border
One to one. Each circle mapped to different square.
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

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Given a function, $f : D \to R$.

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Isomorphism principle:
If there is a bijection $f : D \to R$ then $|D| = |R|$.

Countable.

How to count?

Countable.

How to count?

0,

Countable.

How to count?

0, 1,

Countable.

How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

0, 1, 2, 3, ...

Countable.

How to count?

0, 1, 2, 3, . . .

The Counting numbers.

Countable.

How to count?

0, 1, 2, 3, . . .

The Counting numbers.
The natural numbers! N

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.
The natural numbers! N

Definition:
S is **countable** $\equiv$ bijection between S and some subset of N.

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S is **countable** $\equiv$ bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

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The Counting numbers.
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Definition:
S is **countable** $\equiv$ bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where’s 0?

Which is bigger?

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The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

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More natural numbers!

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But.. but Where's zero? "Comes from 1."

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Bijection from A to $B \implies$ a bijection from B to A.

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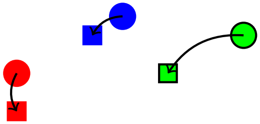

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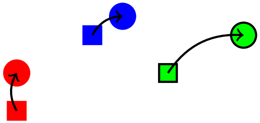

Inverse function!

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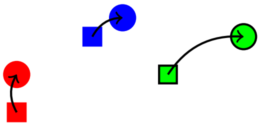

Inverse function!

Can prove equivalence either way.

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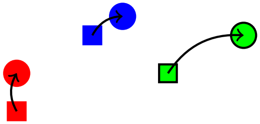

Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

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$f : N \to E$.

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$f : N \to E$.

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Onto:

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Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

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Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

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then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
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All integers?

What about Integers, Z?
Define $f : N \rightarrow Z$.

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Integers and naturals have same size!

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61A

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61A —- streams!

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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Never get to 1.

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Enumerate the rational numbers in order...

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Can't list in "order".

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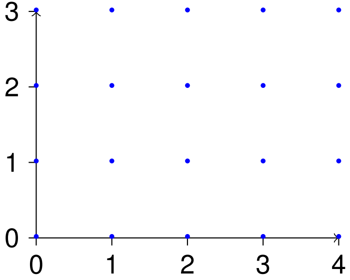

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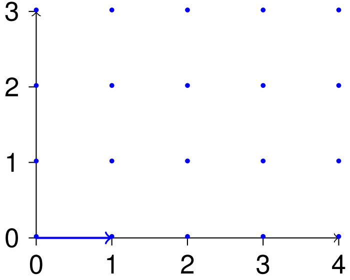

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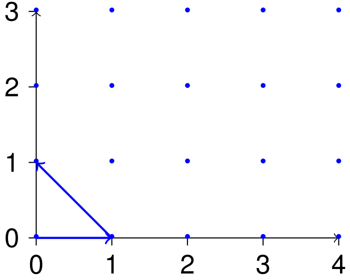

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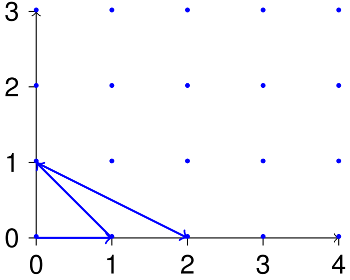

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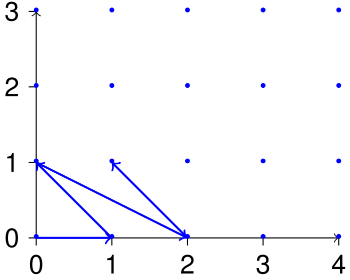

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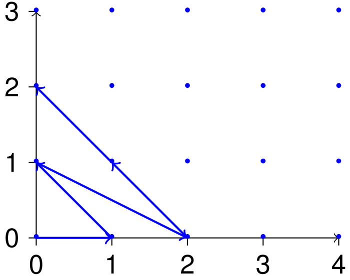

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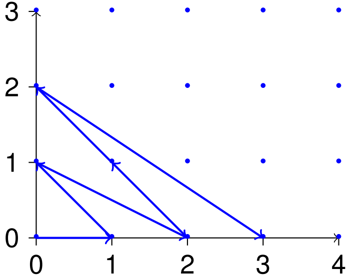

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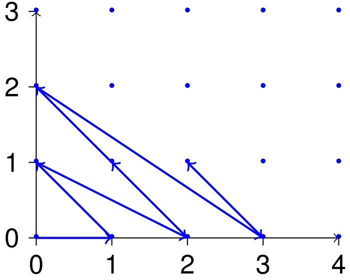

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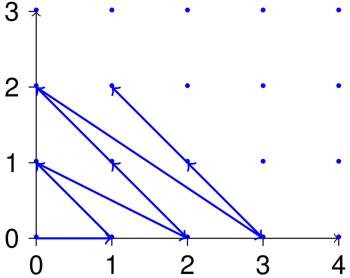

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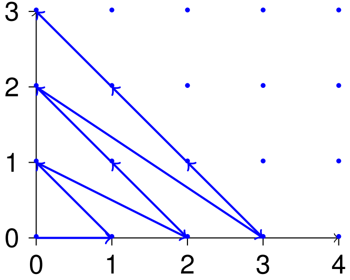

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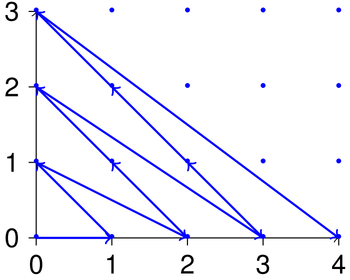

Pairs of natural numbers.

Enumerate in list:
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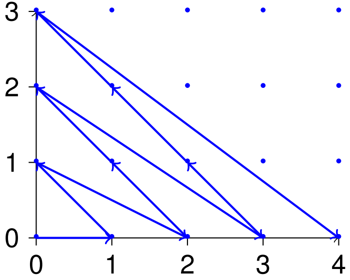

The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!

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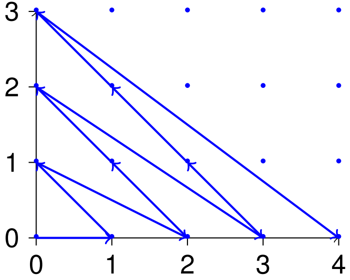

The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!
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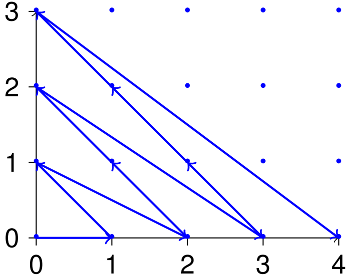

The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!
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Countably infinite.

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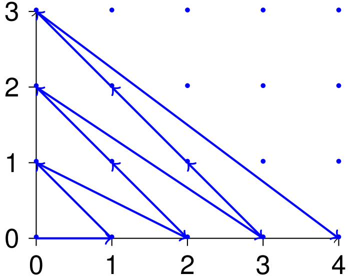

The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!
(i.e., "triangle").

Countably infinite.

Same size as the natural numbers!!

Rationals?

Positive rational number.

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Lowest terms: a/b

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Lowest terms: a/b
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Infinite subset of $N \times N$.

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Interleave Streams in 61A

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The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

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Diagonalization.

If countable, there a listing, L contains all reals.

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0: .500000000...

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Diagonalization.

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1: .785398162...
2: .367879441...

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
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Construct "diagonal" number:

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2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .7

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct “diagonal” number: .77

Diagonalization.

If countable, there a listing, *L* contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .776

Diagonalization.

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2: .367879441...
3: .632120558...
4: .345212312...
⋮

Construct "diagonal" number: .7767

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2: .367879441...
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⋮

Construct "diagonal" number: .77677

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Construct "diagonal" number: $.77677\ldots$

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Diagonal Number:

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Construct "diagonal" number: .77677...

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
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Diagonal number for a list differs from every number in list!

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Diagonal number not in list.

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Contradiction!

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Contradiction!

Set $[0,1]$ is not countable!!

All reals?

Set $[0, 1]$ is not countable!!

All reals?

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What about all reals?

All reals?

Set $[0, 1]$ is not countable!!

What about all reals?
No.

All reals?

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What about all reals?
No.

Any subset of a countable set is countable.

All reals?

Set $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

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3. Use the diagonal from the list to construct a new element t.

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   $\implies$ t is not in the list.

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   $\Longrightarrow$ t is not in the list.
5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,

Another diagonalization.

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evens,

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Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
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Assume is countable.

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes,

Assume is countable.

There is a listing, *L*, that contains all subsets of *N*.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
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Define a diagonal set, D:

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If ith set in L does not contain i, $i \in D$.

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Contradiction.

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Theorem: The set of all subsets of *N* is not countable.

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L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Diagonalize Natural Number.

Natural numbers have a listing, L.

Diagonalize Natural Number.

Natural numbers have a listing, L.

Make a diagonal number, D:
differ from ith element of L in ith digit.

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Any natural number has a finite number of digits.

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"Construction" requires an infinite number of digits.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

Cardinalities of uncountable sets?

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$f : R^+ \rightarrow [0,1]$.

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$[0,1]$ is same cardinality as nonnegative reals!

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

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The powerset of a set, the set of all subsets, is larger than the set.

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Next Topic: Undecidability.

- Undecidability.

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Get around paradox?
The barber lies.

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See Logicomix by Doxiaidis, Papadimitriou (professor here), Papadatos, Di Donna.

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Write me a program checker!

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Run P on I and check!

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Something about infinity here, maybe?

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Theorem: There is no program HALT.

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Proof: Yes!

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Another view of proof: diagonalization.

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P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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Really Same: Says there is no text which can be Turing.