Mark what’s true.
(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) There are more pairs of natural numbers than natural numbers.

Two sets are the same size?
(A) Bijection between the sets.
(B) Count the objects and get the same number. Same size.
(C) Counting to infinity is hard.
(A), (B), (C)?

Given a function, \( f : D \to R \).
One to One:
For all \( x, y \in D, x \neq y \implies f(x) \neq f(y) \).
or
\( \forall x, y \in D, f(x) = f(y) \implies x = y \).
Onto: For all \( y \in R, \exists x \in D, y = f(x) \).
\( f() \) is a bijection if it is one to one and onto.

Isomorphism principle:
If there is a bijection \( f : D \to R \) then \( |D| = |R| \).
Countable.

How to count?
0, 1, 2, 3, ... The Counting numbers.
The natural numbers! N

Definition: S is countable if there is a bijection between S and some subset of N.
If the subset of N is finite, S has finite cardinality.
If the subset of N is infinite, S is countably infinite.

More large sets.

E - Even natural numbers?
f : N → E.

f(n) = 2n.

Onto: ∀ e ∈ E, f(e/2) = e. e/2 is natural since e is even
One-to-one: ∀ x, y ∈ N, x ≠ y → 2x ≠ 2y → f(x) ≠ f(y)

Evens are countably infinite.
Evens are same size as all natural numbers.

Where's 0?

Which is bigger?
The positive integers, Z+, or the natural numbers, N.

Natural numbers. 0, 1, 2, 3, ....
Positive integers. 1, 2, 3, ....

Where's 0?

More natural numbers!
Consider f(z) = z - 1.

For any two z1 ≠ z2 → z1 - 1 ≠ z2 - 1 → f(z1) ≠ f(z2).
One to one!

For any natural number n, for z = n + 1, f(z) = (n + 1) - 1 = n.
Onto for N

Bijection! |Z+| = |N|.

But.. but Where's zero? "Comes from 1."

All integers?

What about integers, Z?

Define f : N → Z.

f(n) = \[ \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if } n \text{ odd} \end{cases} \]

One-to-one: For x ≠ y
if x is even and y is odd, then f(x) is nonnegative and f(y) is negative → f(x) ≠ f(y)
if x is even and y is even, then x/2 ≠ y/2 → f(x) ≠ f(y)

Onto: For any z ∈ Z,
if z ≥ 0, f(2z) = z and 2z ∈ N.
if z < 0, f(2z - 1) = z and 2z + 1 ∈ N.

Integers and naturals have same size!

A bijection is a bijection.

Notice that there is a bijection between N and Z as well.
f(n) = n + 1. 0 → 1, 1 → 2, ...

Bijection from A to B → a bijection from B to A.

Inverse function!
Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

Listings..

<table>
<thead>
<tr>
<th>n</th>
<th>f(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Another View:

<table>
<thead>
<tr>
<th>n</th>
<th>f(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Notice that: A listing "is" a bijection with a subset of natural numbers.
Function = "Position in list."
If finite: bijection with \([0, \ldots, |S| - 1]\)
If infinite: bijection with N.
Enumerability ≡ countability.

Enumerating (listing) a set implies that it is countable.
"Output element of S", "Output next element of S".
Any element x of S has specific, finite position in list.
Z = {0, 1, −1, 2, −2, ...} and then {−1, −2, ...}.
When do you get to −1? at infinity?
Need to be careful.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.
Enumerate T as follows:
Get next element, x, of S, output only if x ∈ T.
Implications:
Z + is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.
All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.
B = {0, 1}∗.
B = {φ, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, ...}.
φ is empty string.
For any string, it appears at some position in the list.
If n bits, it will appear before position 2n+1.
Should be careful here.

Pairs of natural numbers.

Consider pairs of natural numbers: N × N
E.g.: (1, 2), (100, 30), etc.
For finite sets S1 and S2, then S1 × S2 has size |S1| × |S2|.
So, N × N is countably infinite squared ???

Enumerate in list:
(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), ...
0 1 2 3 4
· · · · ... ≈(a + b + 1)(a + b)/2 elements of list!
(i.e., "triangle").
Countably infinite.
Same size as the natural numbers!!
### Poll.

**Enumeration to get bijection with naturals?**

- (A) Integers: First all negatives, then positives.
- (B) Integers: By absolute value, break ties however.
- (C) Pairs of naturals: by sum of values, break ties however.
- (D) Pairs of naturals: by value of first element.
- (E) Pairs of integers: by sum of values, break ties.
- (F) Pairs of integers: by sum of absolute values, break ties.

- (B), (C), (F).

### Rationals?

- Positive rational number.
- Lowest terms: \( \frac{a}{b} \)
- \( a, b \in \mathbb{N} \)
- \( \gcd(a, b) = 1 \).
- Infinite subset of \( \mathbb{N} \times \mathbb{N} \).
- Countably infinite!
- All rational numbers?
- Negative rationals are countable. (Same size as positive rationals.)
- Put all rational numbers in a list.
- First negative, then nonegative ?? No!
- Repeatedly and alternatively take one from each list.
- Interleave Streams in 61A
- The rationals are countably infinite.

### Real numbers..

Real numbers are same size as integers?

### The reals.

Are the set of reals countable?

- Lets consider the reals \([0, 1]\).
- Each real has a decimal representation.
  - .500000000... (1/2)
  - .785398162... \( \pi/4 \)
  - .367879441... \( 1/e \)
  - .632120558... \( 1−1/e \)
  - .345212312... Some real number

### Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: 500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677...

- Diagonal Number: Digit \( i \) is 7 if number \( i \)'s \( i \)th digit is not 7 and 6 otherwise.
- Diagonal number for a list differs from every number in list!
- Diagonal number not in list.
- The rationals are countably infinite.

### All reals?

- Subset \([0, 1]\) is not countable!!
- What about all reals?
- No.
- Any subset of a countable set is countable.
- If reals are countable then so is \([0, 1]\).
Diagonalization.

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$.
3. Use the diagonal from the list to construct a new element $t$.
4. Show that $t$ is different from all elements in the list $\Rightarrow t$ is not in the list.
5. Show that $t$ is in $S$.
6. Contradiction.

Another diagonalization.

The set of all subsets of $N$.
Example subsets of $N$: $(0), (0,\ldots,7)$, evens, odds, primes,
Assume is countable.
There is a listing, $L$, that contains all subsets of $N$.
Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
otherwise $i \notin D$.
$D$ is different from $i$th set in $L$ for every $i$.
$\Rightarrow D$ is not in the listing.
$D$ is a subset of $N$.
$L$ does not contain all subsets of $N$.
Contradiction.
Theorem: The set of all subsets of $N$ is not countable.
(The set of all subsets of $S$, is the powerset of $N$.)

Diagonalize Natural Number.

Natural numbers have a listing, $L$.
Make a diagonal number, $D$:
differ from $i$th element of $L$ in $i$th digit.
Differs from all elements of listing.
$D$ is a natural number... Not.
Any natural number has a finite number of digits.
"Diagonal number construction" requires an infinite number of digits.

Poll: diagonalization Proof.

Mark parts of proof.
(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
(D) Reals can’t be in a list: diagonal number not on list.
(E) Powerset in list: diagonal set not in list.
(B), (C), (D), (E)

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.
First of Hilbert’s problems!

Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?
$f : R^+ \to [0,1]$.
$f(x) =
\begin{cases}
    x + \frac{1}{2} & 0 \leq x \leq \frac{1}{2} \\
    \frac{1}{2x} & x > \frac{1}{2}
\end{cases}
$
One to one. $x \neq y$
If both in $[0,1/2]$, a shift $\Rightarrow f(x) \neq f(y)$.
If neither in $[0,1/2]$ a division $\Rightarrow f(x) \neq f(y)$.
If one is in $[0,1/2]$ and one isn’t, different ranges $\Rightarrow f(x) \neq f(y)$.
Bijection!
$[0,1]$ is same cardinality as nonnegative reals!
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.
The powerset of a set is the set of all subsets.

Resolution of hypothesis?

Gödel, 1940.
Can't use math!
If math doesn't contain a contradiction.
This statement is a lie.
Is the statement above true?
The barber shaves every person who does not shave themselves.
Who shaves the barber?
Self reference.
Can a program refer to a program?
Can a program refer to itself?
Uh oh.....