Refresh: Counting.

First Rule of counting: Objects from a sequence of choices:
- \( n_i \) possibilities for \( i \)th choice.
- \( n_1 \times n_2 \times \cdots \times n_k \) objects.

Second Rule of counting: If order does not matter.
- Count with order. Divide by number of orderings/sorted object.
- Typically: \( \binom{n}{k} \).

Stars and Bars: Sample \( k \) objects with replacement from \( n \).
- Order doesn’t matter. \( k \) stars \( n-1 \) bars.
- Typically: \( \binom{n+k-1}{k} \) or \( \binom{n+k-1}{n-1} \).

Inclusion/Exclusion: two sets of objects.
- Add number of each and then subtract intersection of sets.
- Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.
- Pascal’s Triangle Example: \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \).
- RHS: Number of subsets of \( n+1 \) items size \( k \).
- LHS: \( \binom{n}{k-1} \) counts subsets of \( n+1 \) items with first item.
- \( \binom{n}{k} \) counts subsets of \( n+1 \) items without first item.
- Disjoint – so add!
CS70: On to probability.

Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space
Key Points

▶ Uncertainty does not mean “nothing is known”
▶ How to best make decisions under uncertainty?
  ▶ Buy stocks
  ▶ Detect signals (transmitted bits, speech, images, radar, diseases, etc.)
  ▶ Control systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)
▶ How to best use ‘artificial’ uncertainty?
  ▶ Play games of chance
  ▶ Design randomized algorithms.
▶ Probability
  ▶ Models knowledge about uncertainty
  ▶ Optimizes use of knowledge to make decisions
The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability:
Precise, unambiguous, simple(!) way to reason about uncertainty.

Our mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice on examples and problems.
Flip a fair coin: (One flips or tosses a coin)

- Possible outcomes: Heads ($H$) and Tails ($T$) 
  (One flip yields either ‘heads’ or ‘tails’.)
- Likelihoods: $H : 50\%$ and $T : 50\%$
Random Experiment: Flip one Fair Coin

Flip a fair coin:

What do we mean by the likelihood of tails is 50%?

Two interpretations:

▶ Single coin flip: 50% chance of ‘tails’ [subjectivist]
  *Willingness to bet on the outcome of a single flip*

▶ Many coin flips: About half yield ‘tails’ [frequentist]
  *Makes sense for many flips*

▶ Question: Why does the fraction of tails converge to the same value every time? Statistical Regularity! Deep!
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set $\Omega$ of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$.  

[Diagram showing a coin flip with outcomes H and T, each with a probability of 0.5]
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

- Possible outcomes: Heads ($H$) and Tails ($T$)
- Likelihoods: $H : p \in (0, 1)$ and $T : 1 - p$
- Frequentist Interpretation:
  
  Flip many times $\Rightarrow$ Fraction $1 - p$ of tails

- Question: How can one figure out $p$? Flip many times

- Tautology? No: Statistical regularity!
Random Experiment: Flip one Unfair Coin

Flip an **unfair** (biased, loaded) coin: model

Physical Experiment

Probability Model

\[ \Omega \]
\[ H \circ p \]
\[ T \circ (1 - p) \]
Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\} \equiv \{H, T\}^2$.
- Note: $A \times B := \{(a, b) \mid a \in A, b \in B\}$ and $A^2 := A \times A$.
- Likelihoods: $1/4$ each.
Flip Glued Coins

Flips two coins glued together side by side:

- **Possible outcomes:** $\{HT, TH\}$.
- **Likelihoods:** $HT : 0.5$, $TH : 0.5$.
- **Note:** Coins are glued so that they show different faces.
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: \{HH, HT, TH, TT\}.
- Likelihoods: HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4.
- Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Here is a way to summarize the four random experiments:

- $\Omega$ is the set of possible outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are $\geq 0$ and add up to 1;
- Fair coins: [1]; Glued coins: [3], [4];
- Spring-attached coins: [2];
Flipping Two Coins

Important remarks:

▶ Each outcome describes the two coins.
▶ E.g., $HT$ is one outcome of each of the above experiments.
▶ **Wrong** to think that outcomes are $\{H, T\}$ and that one picks twice from that set.
▶ Indeed, this viewpoint misses the relationship between the two flips.
▶ Each $\omega \in \Omega$ describes one outcome of the complete experiment.
▶ $\Omega$ and the probabilities specify the random experiment.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

- Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$. Thus, $2^n$ possible outcomes.
- Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H, T\}^n$.
- $A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.
- Likelihoods: $1/2^n$ each.
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes: \( \{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\} \).
- Likelihoods: \( \frac{1}{36} \) for each.
Probability Space.

1. A “random experiment”:
   (a) Flip a biased coin;
   (b) Flip two fair coins;
   (c) Deal a poker hand.

2. A set of possible outcomes: $\Omega$.
   (a) $\Omega = \{H, T\}$;
   (b) $\Omega = \{HH, HT, TH, TT\}; \ |\Omega| = 4$;
   (c) $\Omega = \{A♠ A♦ A♣ A♥ K♠, A♠ A♦ A♣ A♥ Q♠, \ldots\}$
   \(|\Omega| = (\binom{52}{5})\).

3. Assign a probability to each outcome: $Pr : \Omega \rightarrow [0, 1]$.
   (a) $Pr[H] = p, Pr[T] = 1 - p$ for some $p \in [0, 1]$;
   (b) $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$
   (c) $Pr[ A♠ A♦ A♣ A♥ K♠ ] = \ldots = 1 / (\binom{52}{5})$
Probability Space: formalism.

\( \Omega \) is the **sample space**. 
\( \omega \in \Omega \) is a **sample point**. (Also called an **outcome**.)
Sample point \( \omega \) has a probability \( Pr[\omega] \) where

- \( 0 \leq Pr[\omega] \leq 1 \);
- \( \sum_{\omega \in \Omega} Pr[\omega] = 1 \).
Probability Space: Formalism.

In a **uniform probability space** each outcome \( \omega \) is equally probable: 
\[
Pr[\omega] = \frac{1}{|\Omega|} \quad \text{for all } \omega \in \Omega.
\]

Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.
Probability Space: Formalism

Simplest physical model of a **uniform** probability space:

A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

$$\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$$

$$Pr[\text{blue}] = \frac{1}{8}.$$
Probability Space: Formalism

Simplest physical model of a **non-uniform** probability space:

\[ Ω = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \frac{3}{10}, \quad Pr[\text{Green}] = \frac{4}{10}, \quad \text{etc.} \]

Note: Probabilities are restricted to rational numbers: \( \frac{N_k}{N} \).
Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$$\Omega = \{1, 2, 3, \ldots, N\}, Pr[\omega] = p_\omega.$$
An important remark

- The random experiment selects **one and only one** outcome in $\Omega$.
- For instance, when we flip a fair coin **twice**
  - $\Omega = \{HH, TH, HT, TT\}$
  - The experiment selects **one** of the elements of $\Omega$.
- In this case, it's wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets $HH$ or $TT$ with probability 50% each. This is not captured by ‘picking two outcomes.’
Summary of Probability Basics

1. Random Experiment

2. Probability Space: $\Omega; Pr[\omega] \in [0, 1]; \sum_\omega Pr[\omega] = 1$.

3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$. 

Modeling Uncertainty: Probability Space
Onwards in Probability.

Events, Conditional Probability, Independence, Bayes’ Rule
Today: Events.
Probability Basics Review

Setup:

▶ Random Experiment.
  Flip a fair coin twice.

▶ Probability Space.

  ▶ **Sample Space:** Set of outcomes, $\Omega$.
    $\Omega = \{HH, HT, TH, TT\}$
    (Note: Not $\Omega = \{H, T\}$ with two picks!)

  ▶ **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
    $Pr[HH] = \cdots = Pr[TT] = 1/4$
    1. $0 \leq Pr[\omega] \leq 1$.
    2. $\sum_{\omega \in \Omega} Pr[\omega] = 1.$
Set notation review

- **Figure: Two events**
- **Figure: Union (or)**
- **Figure: Difference (A, not B)**
- **Figure: Complement (not)**
- **Figure: Intersection (and)**
- **Figure: Symmetric difference (only one)**
Probability of exactly one ‘heads’ in two coin flips?

Idea: Sum the probabilities of all the different outcomes that have exactly one ‘heads’: \( HT, TH. \)

This leads to a definition!

**Definition:**

- An **event**, \( E \), is a subset of outcomes: \( E \subset \Omega. \)

- The **probability** of \( E \) is defined as \( Pr[E] = \sum_{\omega \in E} Pr[\omega]. \)
Event: Example

Physical experiment

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]
\[ Pr[\text{Red}] = \frac{3}{10}, \quad Pr[\text{Green}] = \frac{4}{10}, \quad \text{etc.} \]

\[ E = \{ \text{Red, Green} \} \Rightarrow Pr[E] = \frac{3 + 4}{10} = \frac{3}{10} + \frac{4}{10} = Pr[\text{Red}] + Pr[\text{Green}]. \]
Probability of exactly one heads in two coin flips?

Sample Space, \( \Omega = \{ HH, HT, TH, TT \} \).

Uniform probability space: \( Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4} \).

Event, \( E \), “exactly one heads”: \( \{ TH, HT \} \).

\[
Pr[E] = \sum_{\omega \in E} Pr[\omega] = \frac{|E|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}
\]
Roll a red and a blue die.

\[
Pr[\text{Sum to 7}] = \frac{6}{36} \quad Pr[\text{Sum to 10}] = \frac{3}{36}
\]
**Example: 20 coin tosses.**

20 coin tosses

Sample space: \( \Omega = \) set of 20 fair coin tosses.
\( \Omega = \{ T, H \}^{20} \equiv \{ 0, 1 \}^{20}; \quad |\Omega| = 2^{20}. \)

What is more likely?

\( \omega_1 := (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \) or
\( \omega_2 := (1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0)? \)

Answer: Both are equally likely: \( Pr[\omega_1] = Pr[\omega_2] = \frac{1}{|\Omega|}. \)

What is more likely?

\( (E_1) \) Twenty Hs out of twenty, or
\( (E_2) \) Ten Hs out of twenty?

Answer: Ten Hs out of twenty.

Why? There are many sequences of 20 tosses with ten Hs; only one with twenty Hs. \( \Rightarrow Pr[E_1] = \frac{1}{|\Omega|} \ll Pr[E_2] = \frac{|E_2|}{|\Omega|}. \)

\(|E_2| = \binom{20}{10} = 184,756. \)
Probability of \( n \) heads in 100 coin tosses.

\[
\Omega = \{H, T\}^{100}; \quad |\Omega| = 2^{100}.
\]

Event \( E_n = \text{‘} n \text{ heads’} \); \( |E_n| = \binom{100}{n} \)

\[ p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}} \]

Observe:

- Concentration around mean: Law of Large Numbers;
- Bell-shape: Central Limit Theorem.
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$. 
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ “100 coin tosses with exactly 50 heads”

$|E|$?

Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$
Calculation.

Stirling formula (for large $n$):

\[
n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.
\]

\[
\binom{2n}{n} \approx \frac{\sqrt{4\pi n(2n/e)^{2n}}}{[\sqrt{2\pi n(n/e)^n}]^2} \approx \frac{4^n}{\sqrt{\pi n}}.
\]

\[
Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{50\pi}} \approx .08.
\]
Exactly 50 heads in 100 coin tosses.

\[ Pr[n \text{Hs out of } 2n] = \frac{\binom{2n}{n}}{2^{2n}} \]
Summary.

1. Random Experiment

2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_\omega Pr[\omega] = 1$.

3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.

4. Event: “subset of outcomes.” $A \subseteq \Omega$. $Pr[A] = \sum_{w \in A} Pr[\omega]$

5. Some calculations.