Today

Total Probability: Intuition, pictures, inference.

Bayes Rule.

Balls in Bins.

Birthday Paradox

Coupon Collector

Independence

Definition: Two events A and B are **independent** if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.
- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right)\left(\frac{1}{6}\right)$.
- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$.
- When throwing 3 balls into 3 bins, $A =$ bin 1 is empty and $B =$ bin 2 is empty are not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A]Pr[B] = \left(\frac{8}{27}\right)\left(\frac{8}{27}\right)$.

Independence and conditional probability

Fact: Two events A and B are **independent** if and only if

$$Pr[A|B] = Pr[A].$$

Indeed: $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$, so that

$$Pr[A|B] = Pr[A] \Leftrightarrow \frac{Pr[A \cap B]}{Pr[B]} = Pr[A] \Leftrightarrow Pr[A \cap B] = Pr[A]Pr[B].$$

Causality vs. Correlation

Events A and B are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

A and B being positively correlated does not mean that A causes B or that B causes A.

Other examples:

- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
- Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?

Proving Causality

Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

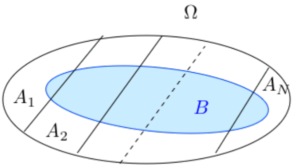
Some difficulties:

- A and B may be positively correlated because they have a common cause. (E.g., being a rabbit.)
- If B precedes A, then B is more likely to be the cause. (E.g., smoking.) However, they could have a common cause that induces B before A. (E.g., smart, CS70, Tesla.)

More about such questions later. For fun, check "N. Taleb: Fooled by randomness."

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



Then,

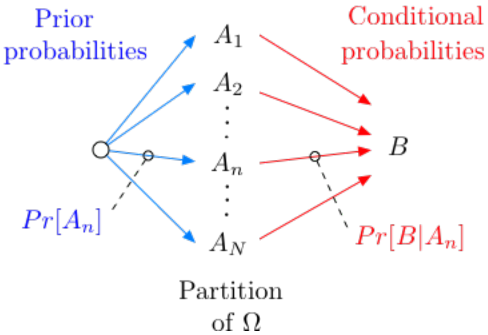
$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, B is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$. Thus,

$$Pr[B] = Pr[A_1]Pr[B|A_1] + \cdots + Pr[A_N]Pr[B|A_N].$$

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



$$Pr[B] = Pr[A_1]Pr[B|A_1] + \cdots + Pr[A_N]Pr[B|A_N].$$

Is you coin loaded?

Your coin is fair w.p. $1/2$ or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

Analysis:

$$A = \text{ 'coin is fair'}, B = \text{ 'outcome is heads'}$$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2, P[B|\bar{A}] = 0.6, Pr[A] = 1/2 = Pr[\bar{A}]$

Now,

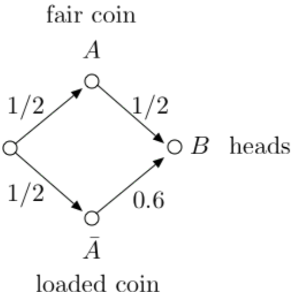
$$\begin{aligned} Pr[B] &= Pr[A \cap B] + Pr[\bar{A} \cap B] = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \\ &= (1/2)(1/2) + (1/2)0.6 = 0.55. \end{aligned}$$

Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

Is you coin loaded?

A picture:



Imagine 100 situations, among which
$m := 100(1/2)(1/2)$ are such that A and B occur and
$n := 100(1/2)(0.6)$ are such that $\bar{A}$ and B occur.

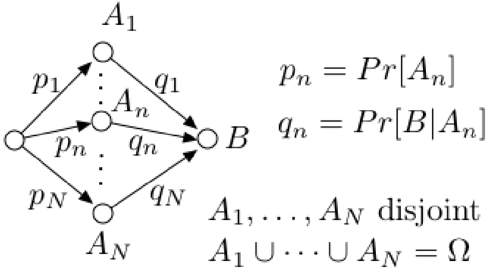
Thus, among the $m+n$ situations where B occurred, there are m where A occurred.

Hence,

$$Pr[A|B] = \frac{m}{m+n} = \frac{(1/2)(1/2)}{(1/2)(1/2)+(1/2)0.6}.$$

Bayes Rule

A general picture: We imagine that there are N possible causes $A_1, \ldots, A_N$.



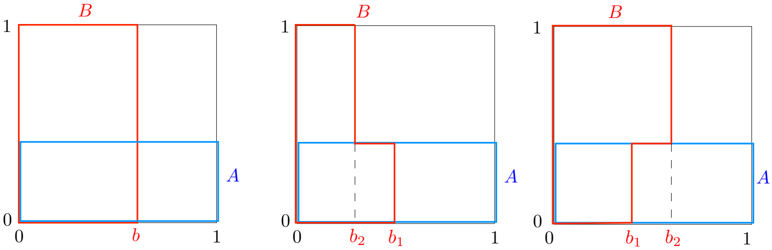
Imagine 100 situations, among which $100 p_n q_n$ are such that A_n and B occur, for $n = 1, \ldots, N$.
Thus, among the $100 \sum_m p_m q_m$ situations where B occurred, there are $100 p_n q_n$ where A_n occurred.

Hence,

$$Pr[A_n|B] = \frac{p_n q_n}{\sum_m p_m q_m}.$$

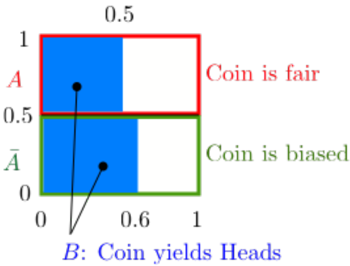
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square



- Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
- Middle: A and B are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.
- Right: A and B are negatively correlated.
  $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_1, b_2)$.

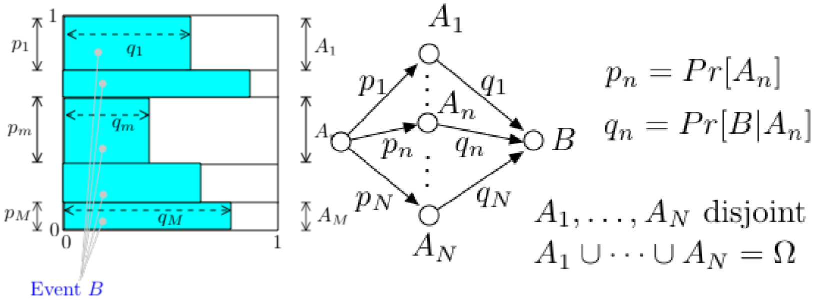
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5$$
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$
$$\approx 0.46 = \text{fraction of } B \text{ that is inside } A$$

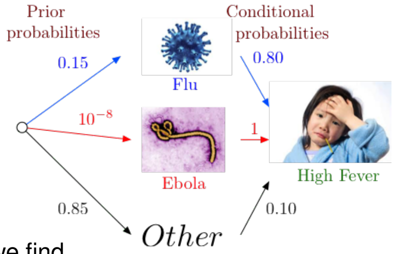
Bayes: General Case



Pick a point uniformly at random in the unit square. Then

$$Pr[A_n] = p_n, n = 1, \ldots, N$$
$$Pr[B|A_n] = q_n, n = 1, \ldots, N; Pr[A_n \cap B] = p_n q_n$$
$$Pr[B] = p_1 q_1 + \cdots p_N q_N$$
$$Pr[A_n|B] = \frac{p_n q_n}{p_1 q_1 + \cdots p_N q_N} = \text{ fraction of } B \text{ inside } A_n.$$

Why do you have a fever?



Using Bayes' rule, we find

$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

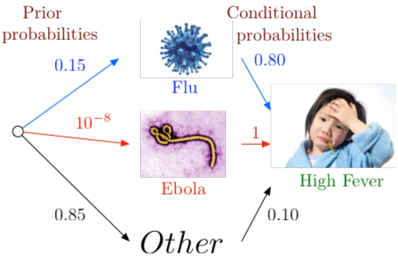
$$Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}$$

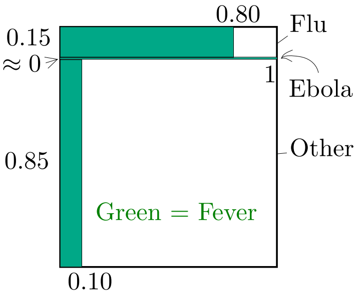
$$Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42$$

The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

Why do you have a fever?

Our “Bayes’ Square” picture:





$$58\% \text{ of Fever} = \text{Flu}$$
$$\approx 0\% \text{ of Fever} = \text{Ebola}$$
$$42\% \text{ of Fever} = \text{Other}$$

Note that even though $Pr[\text{Fever}|\text{Ebola}] = 1$, one has

$$Pr[\text{Ebola}|\text{Fever}] \approx 0.$$

This example shows the importance of the prior probabilities.

Why do you have a fever?

We found

$$\begin{aligned} &Pr[\text{Flu}|\text{High Fever}] \approx 0.58, \\ &Pr[\text{Ebola}|\text{High Fever}] \approx 5 \times 10^{-8}, \\ &Pr[\text{Other}|\text{High Fever}] \approx 0.42 \end{aligned}$$

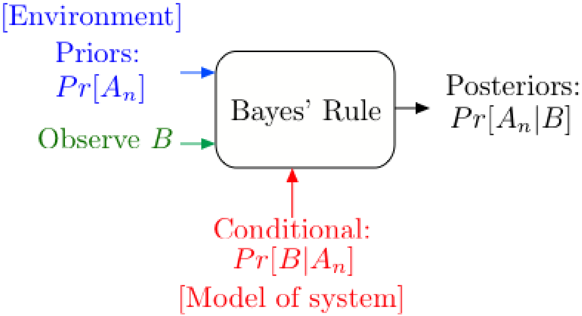
One says that 'Flu' is the Most Likely a Posteriori (MAP) cause of the high fever.
'Ebola' is the Maximum Likelihood Estimate (MLE) of the cause: it causes the fever with the largest probability.
Recall that

$$p_m = Pr[A_m], q_m = Pr[B|A_m], Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots + p_M q_M}.$$

Thus,

- MAP = value of m that maximizes $p_m q_m$.
- MLE = value of m that maximizes q_m.

Bayes' Rule Operations



Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

Fig. 3. *Joshua Bayes (1671–1746).*

$$\xrightarrow{Pr[B|A]}$$

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Random Experiment: Pick a random male.
Outcomes: $(test, disease)$
A - prostate cancer.
B - positive PSA test.

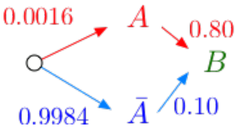
- $Pr[A] = 0.0016$, (.16 % of the male population is affected.)
- $Pr[B|A] = 0.80$ (80% chance of positive test with disease.)
- $Pr[B|\overline{A}] = 0.10$ (10% chance of positive test without disease.)

From http://www.cpcn.org/01_psa_tests.htm and http://seer.cancer.gov/statfacts/html/prost.html (10/12/2011.)

Positive PSA test (B). Do I have disease?

$$Pr[A|B]???$$

Bayes Rule.



Using Bayes' rule, we find

$$P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = .013.$$

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...

Incontinence..

Death.

Quick Review

Events, Conditional Probability, Independence, Bayes' Rule

Key Ideas:

- Conditional Probability:

  $$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$$

- Independence: $Pr[A \cap B] = Pr[A]Pr[B]$.
- Bayes' Rule:

  $$Pr[A_n|B] = \frac{Pr[A_n]Pr[B|A_n]}{\sum_m Pr[A_m]Pr[B|A_m]}.$$

  $Pr[A_n|B] =$ posterior probability; $Pr[A_n] =$ prior probability .

- All these are possible:

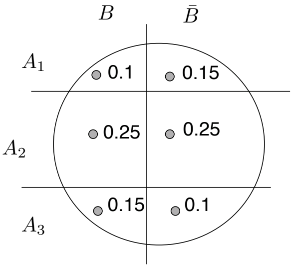
  $$Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A].$$

Independence

Recall :

$$
\begin{aligned}
& A \text{ and } B \text{ are independent} \\
& \Leftrightarrow Pr[A \cap B] = Pr[A]Pr[B] \\
& \Leftrightarrow Pr[A|B] = Pr[A].
\end{aligned}
$$

Consider the example below:

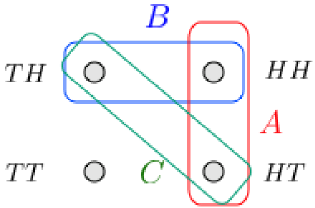


(A_2, B) are independent: $Pr[A_2|B] = 0.5 = Pr[A_2]$.
$(A_2, \bar{B})$ are independent: $Pr[A_2|\bar{B}] = 0.5 = Pr[A_2]$.
(A_1, B) are not independent: $Pr[A_1|B] = \frac{0.1}{0.5} = 0.2 \neq Pr[A_1] = 0.25$.

Pairwise Independence

Flip two fair coins. Let

- $A =$ 'first coin is H' $= \{HT, HH\}$;
- $B =$ 'second coin is H' $= \{TH, HH\}$;
- $C =$ 'the two coins are different' $= \{TH, HT\}$.



A, C are independent; B, C are independent;

$A \cap B, C$ are not independent. ($Pr[A \cap B \cap C] = 0 \neq Pr[A \cap B]Pr[C]$.)

If A did not say anything about C and B did not say anything about C, then $A \cap B$ would not say anything about C.

Example 2

Flip a fair coin 5 times. Let A_n = 'coin n is H', for $n = 1, \ldots, 5$.

Then,

$$A_m, A_n \text{ are independent for all } m \neq n.$$

Also,

$$A_1 \text{ and } A_3 \cap A_5 \text{ are independent.}$$

Indeed,

$$Pr[A_1 \cap (A_3 \cap A_5)] = \frac{1}{8} = Pr[A_1]Pr[A_3 \cap A_5].$$

Similarly,

$$A_1 \cap A_2 \text{ and } A_3 \cap A_4 \cap A_5 \text{ are independent.}$$

This leads to a definition ....

Mutual Independence

Definition Mutual Independence

(a) The events $A_1, \ldots, A_5$ are mutually independent if

$$Pr[\cap_{k \in K} A_k] = \Pi_{k \in K} Pr[A_k], \text{ for all } K \subseteq \{1, \ldots, 5\}.$$

(b) More generally, the events $\{A_j, j \in J\}$ are mutually independent if

$$Pr[\cap_{k \in K} A_k] = \Pi_{k \in K} Pr[A_k], \text{ for all finite} K \subseteq J.$$

Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

Mutual Independence

Theorem

(a) If the events $\{A_j, j \in J\}$ are mutually independent and if K_1 and K_2 are disjoint finite subsets of J, then

$$\cap_{k \in K_1} A_k \text{ and } \cap_{k \in K_2} A_k \text{ are independent.}$$

(b) More generally, if the K_n are pairwise disjoint finite subsets of J, then the events

$$\cap_{k \in K_n} A_k \text{ are mutually independent.}$$

(c) Also, the same is true if we replace some of the A_k by $\bar{A}_k$.

Proof:
See Notes 25, 2.7. $\square$