Today

Total Probability: Intuition, pictures, inference.

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Bayes Rule.

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Balls in Bins.

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Coupon Collector

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- Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?

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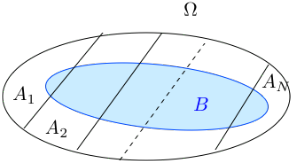
More about such questions later. For fun, check "N. Taleb: Fooled by randomness."

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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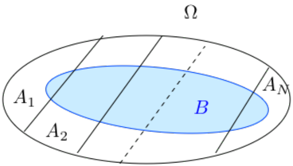

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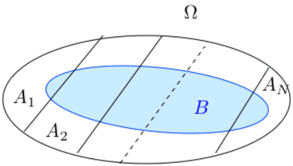

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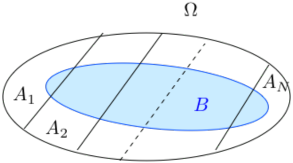

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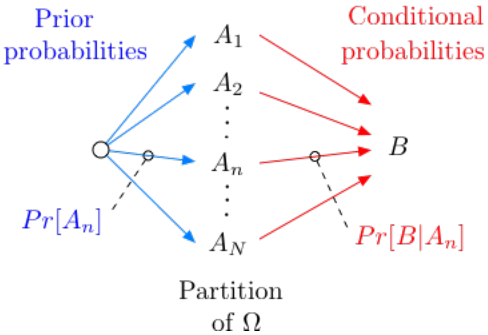

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Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

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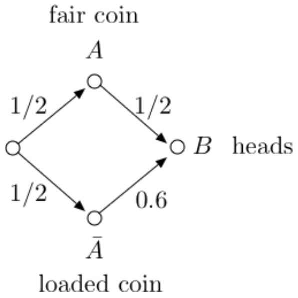

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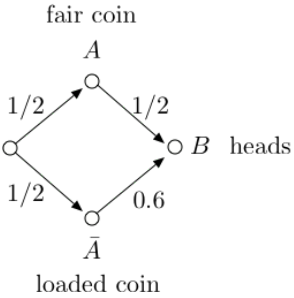

Imagine 100 situations, among which
$m := 100(1/2)(1/2)$ are such that A and B occur and
$n := 100(1/2)(0.6)$ are such that $\bar{A}$ and B occur.

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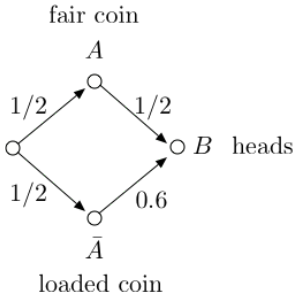
A picture:



Imagine 100 situations, among which
$m := 100(1/2)(1/2)$ are such that A and B occur and
$n := 100(1/2)(0.6)$ are such that $\bar{A}$ and B occur.

Thus, among the $m + n$ situations where B occurred, there are m where A occurred.

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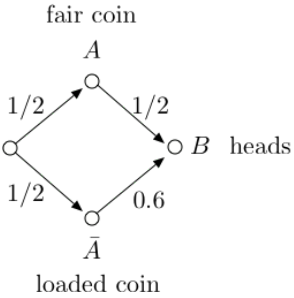

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Hence,

$$Pr[A|B] = \frac{m}{m+n} = \frac{(1/2)(1/2)}{(1/2)(1/2)+(1/2)0.6}.$$

Bayes Rule

A general picture: We imagine that there are N possible causes $A_1, \ldots, A_N$.

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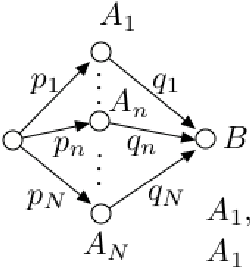

$$p_n = Pr[A_n]$$

$$q_n = Pr[B|A_n]$$

$A_1, \ldots, A_N$ disjoint

$A_1 \cup \cdots \cup A_N = \Omega$

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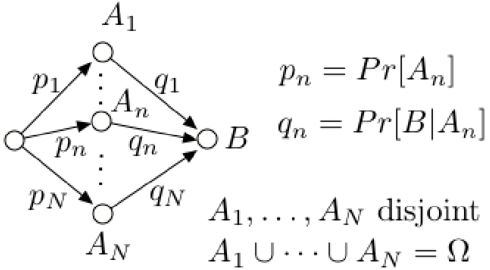

Imagine 100 situations, among which $100 p_n q_n$ are such that A_n and B occur, for $n = 1, \ldots, N$.
Thus, among the $100 \sum_m p_m q_m$ situations where B occurred, there are $100 p_n q_n$ where A_n occurred.

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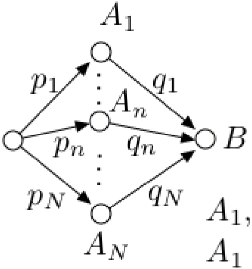

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Conditional Probability: Pictures

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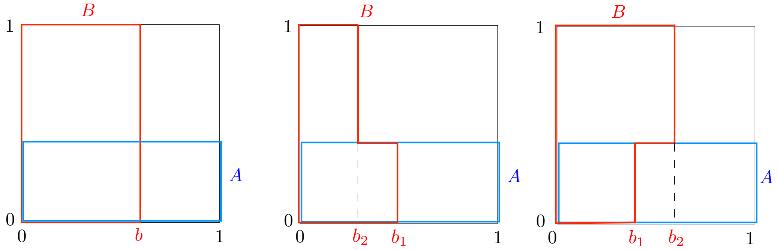
Illustrations: Pick a point uniformly in the unit square



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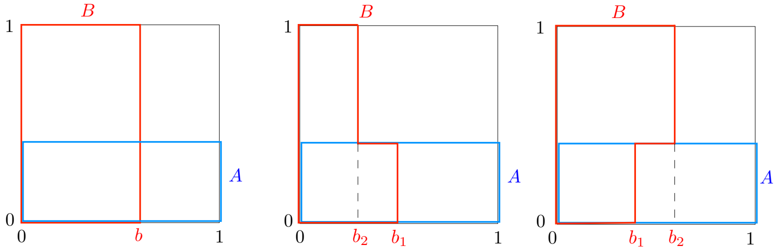

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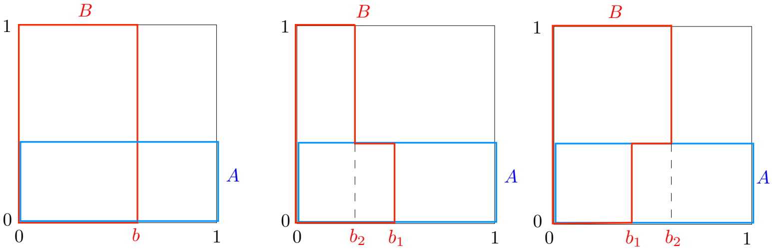

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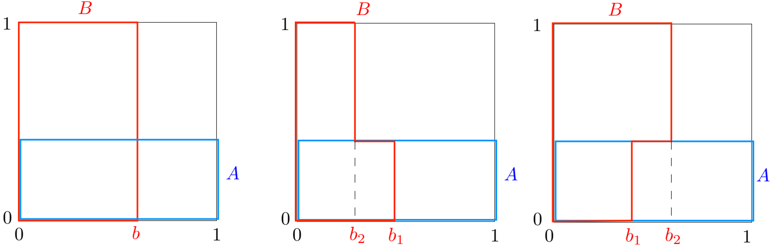

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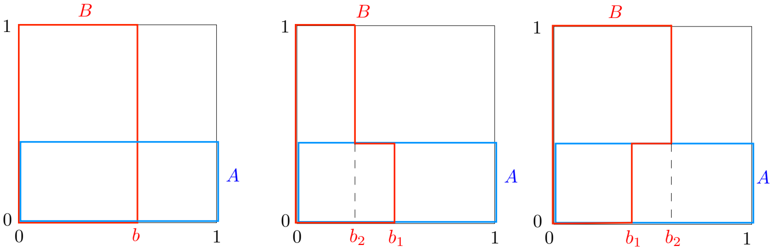

- Left: *A* and *B* are independent. $Pr[B] = b$;

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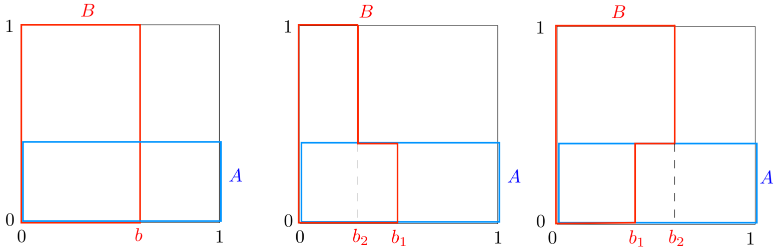

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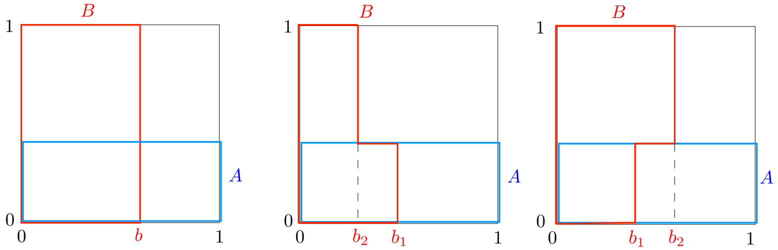

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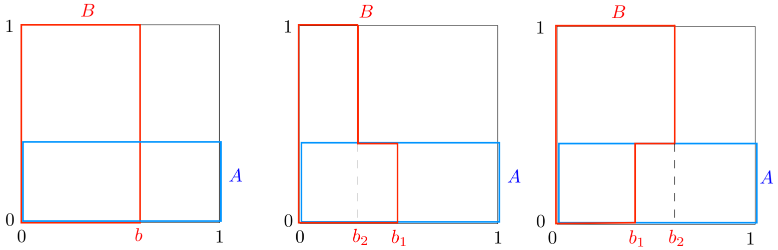

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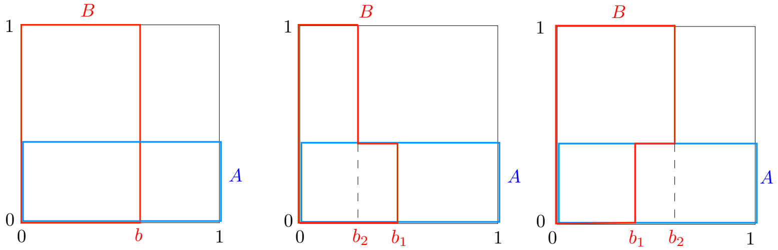

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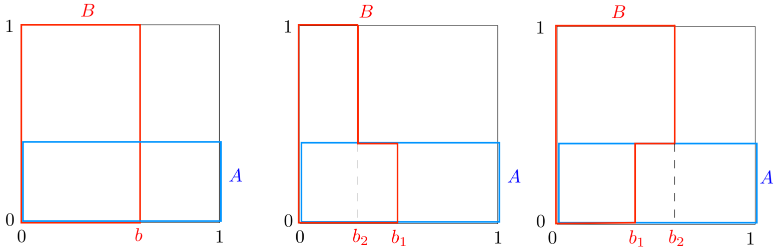

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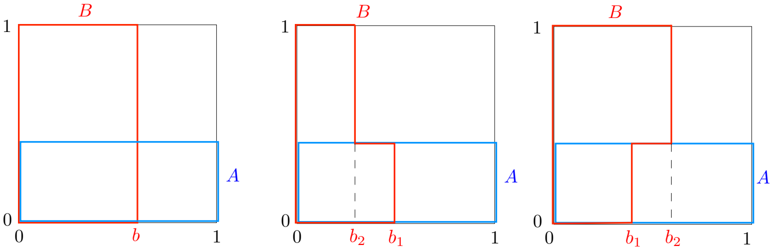

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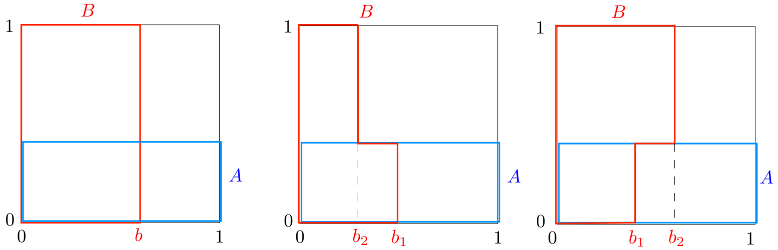

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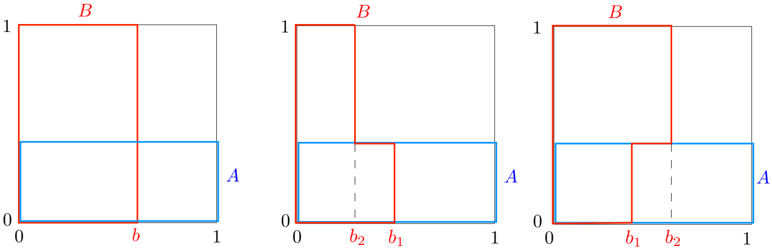

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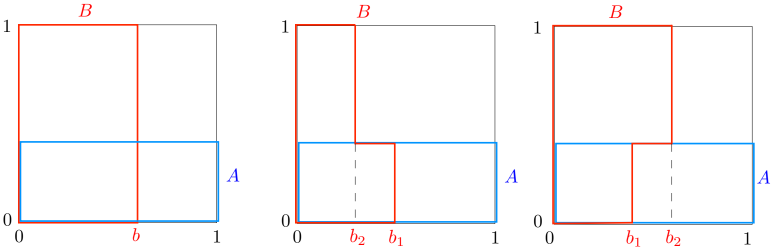

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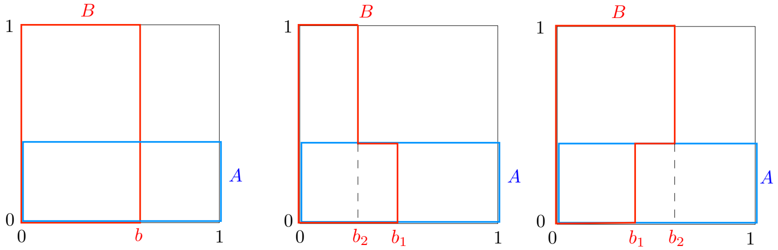

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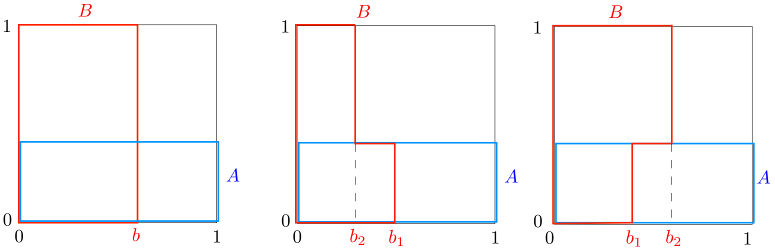

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  $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$.

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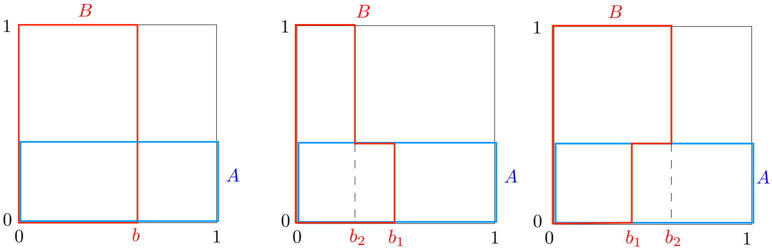

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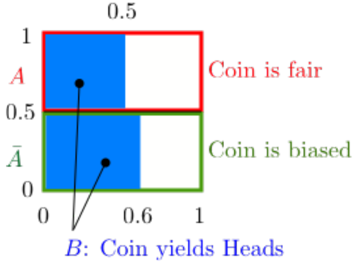
Bayes and Biased Coin

Bayes and Biased Coin



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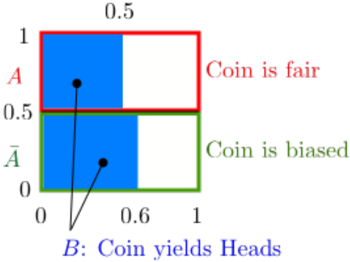

Pick a point uniformly at random in the unit square. Then

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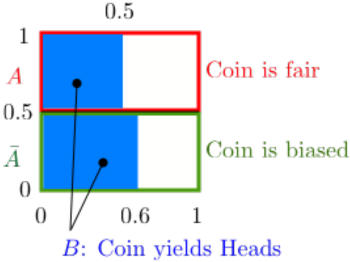

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$Pr[A] =$

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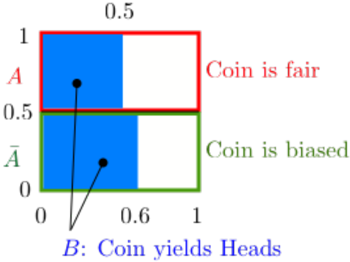

Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5;$

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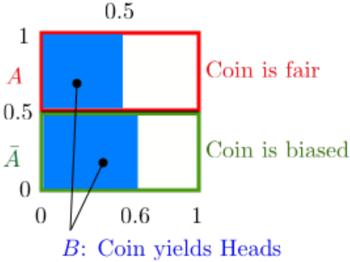

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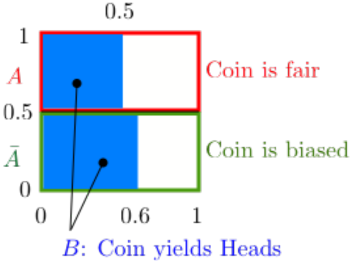
$Pr[A] = 0.5; Pr[\bar{A}] =$

Bayes and Biased Coin



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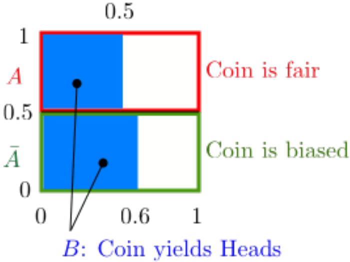

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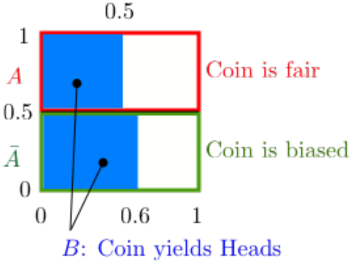

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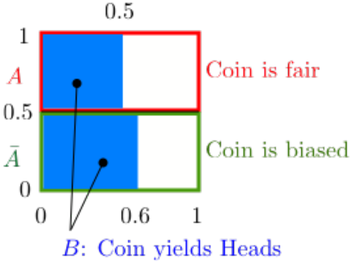

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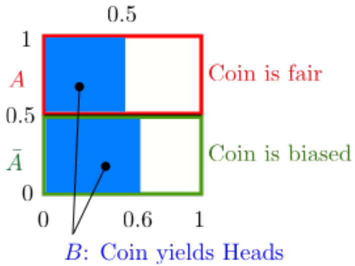
Bayes and Biased Coin



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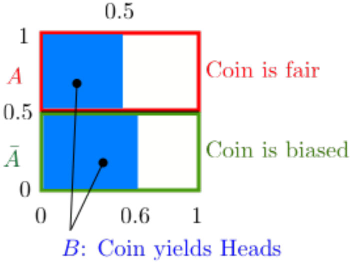
Bayes and Biased Coin



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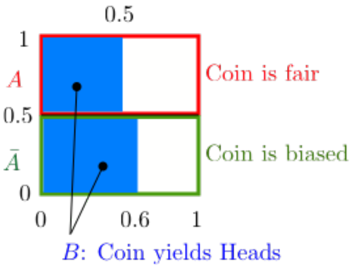

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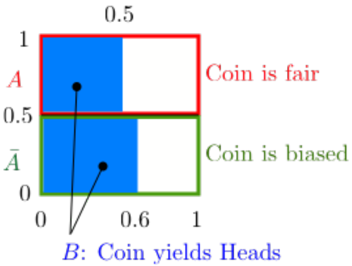

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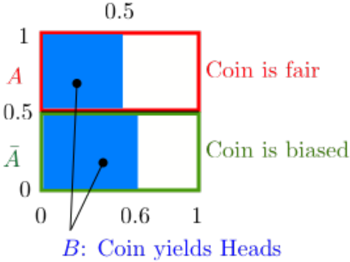

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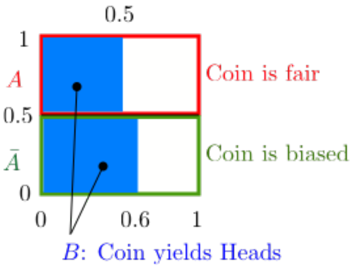

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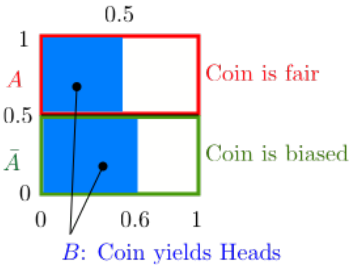
Bayes and Biased Coin



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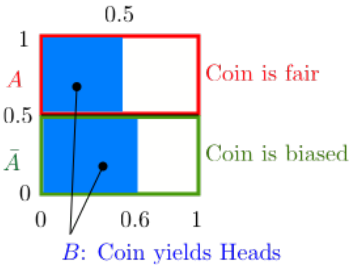
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6}$$

Bayes and Biased Coin



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$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$

$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$

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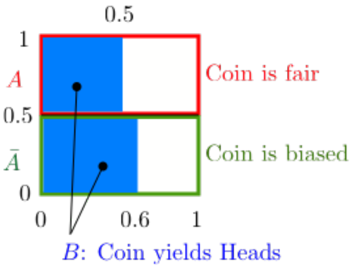

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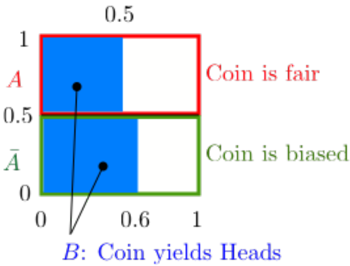
$$\approx 0.46$$

Bayes and Biased Coin



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$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$

$$\approx 0.46 = \text{fraction of } B \text{ that is inside } A$$

Bayes: General Case

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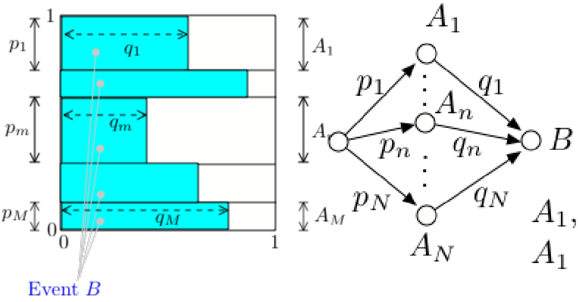

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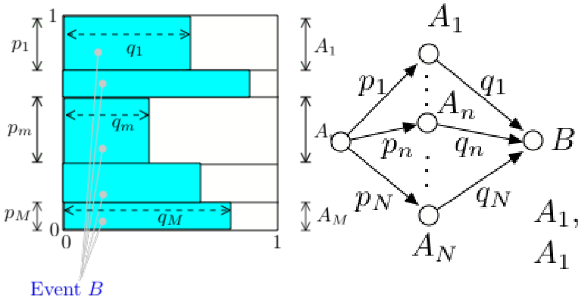

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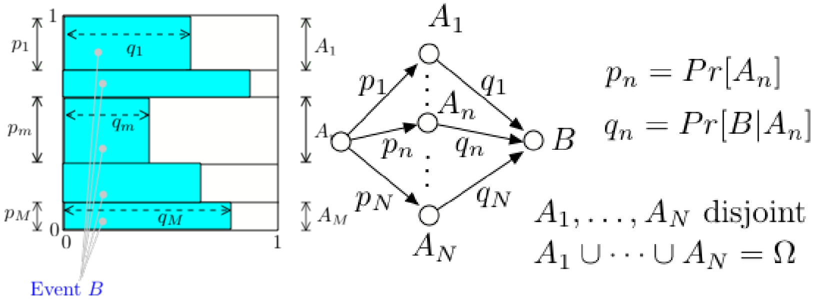

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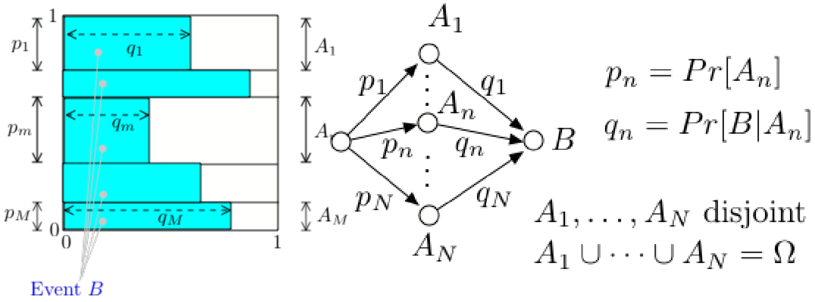

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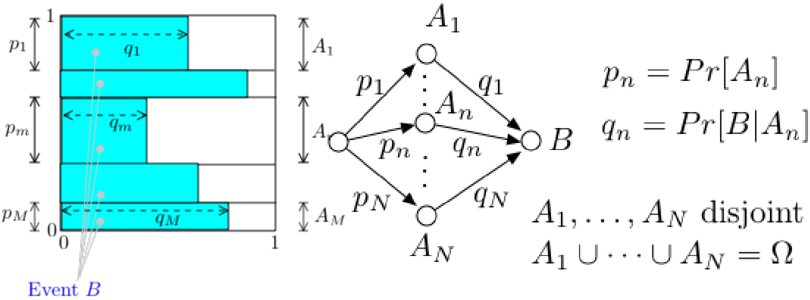

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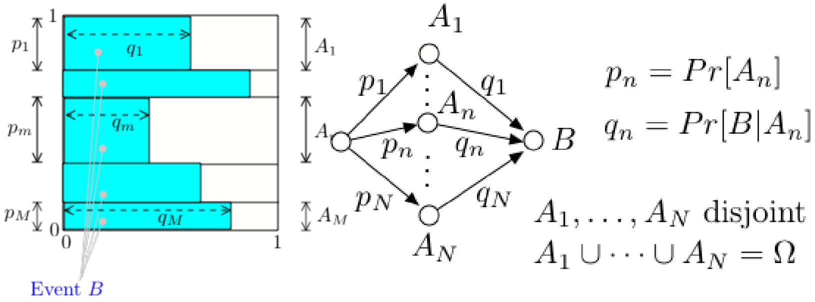

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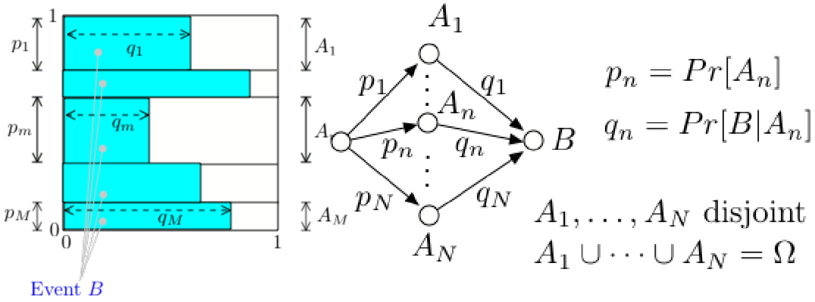

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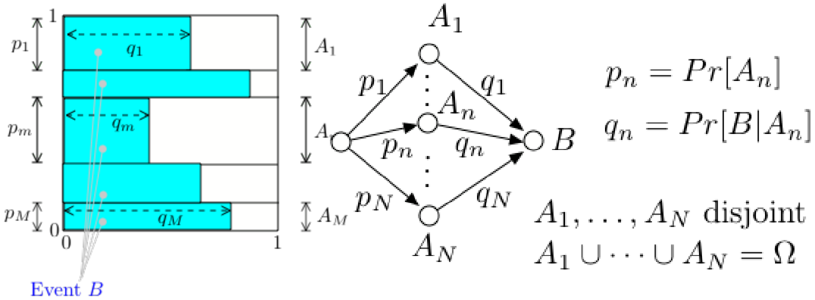

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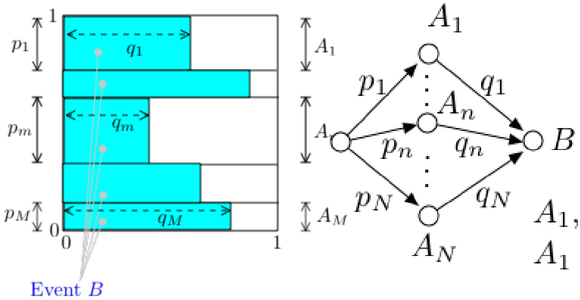

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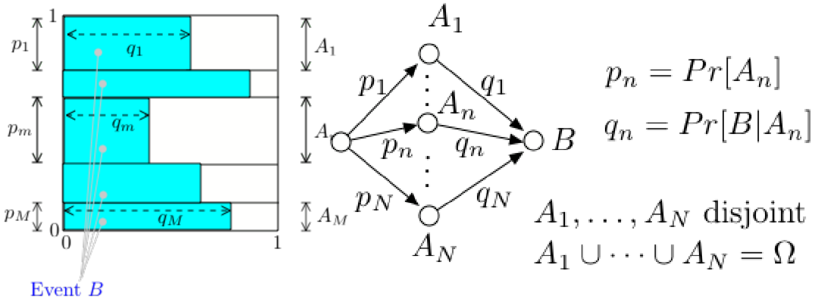

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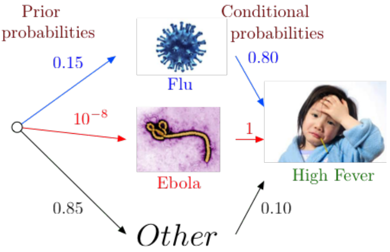

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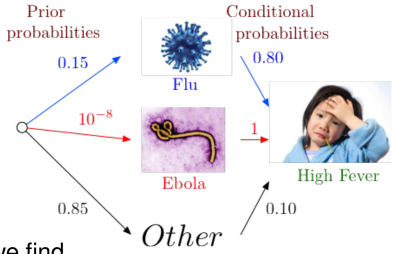

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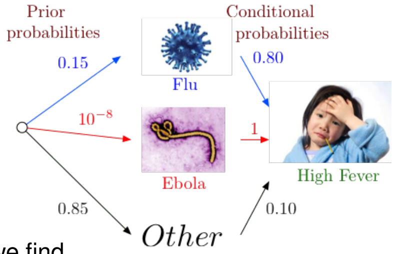

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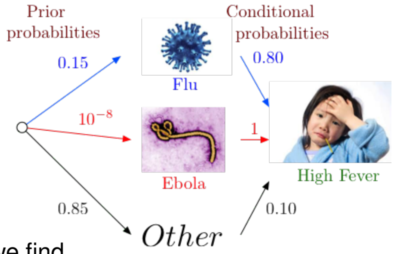

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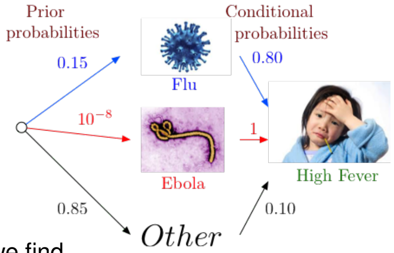

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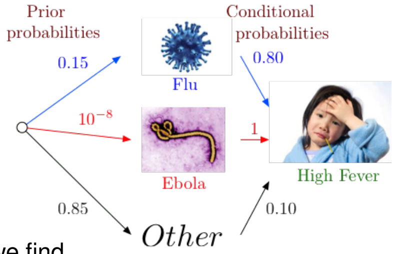

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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

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Our “Bayes’ Square” picture:

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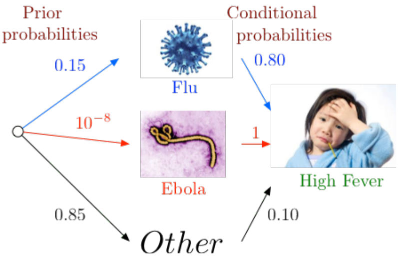

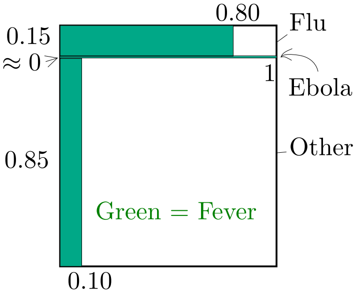

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≈ 0% of Fever = Ebola

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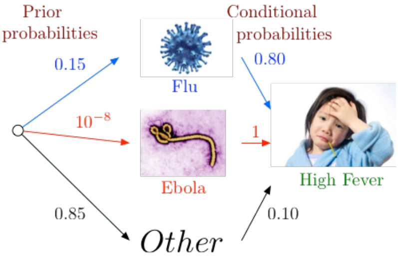

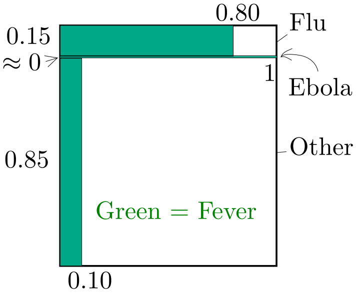

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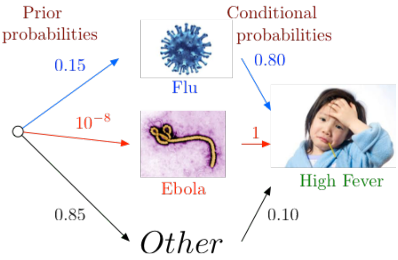

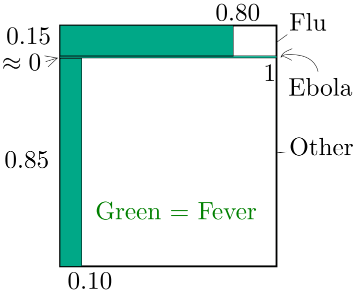

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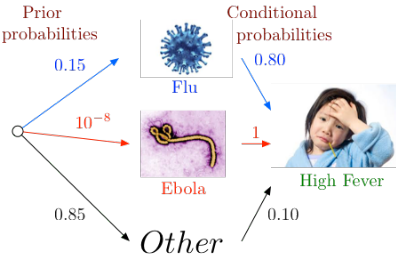

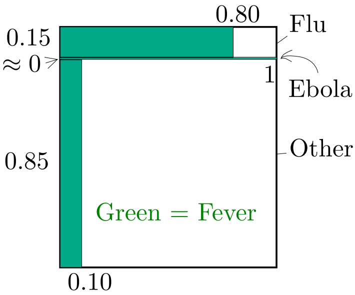

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This example shows the importance of the prior probabilities.

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Bayes' Rule Operations

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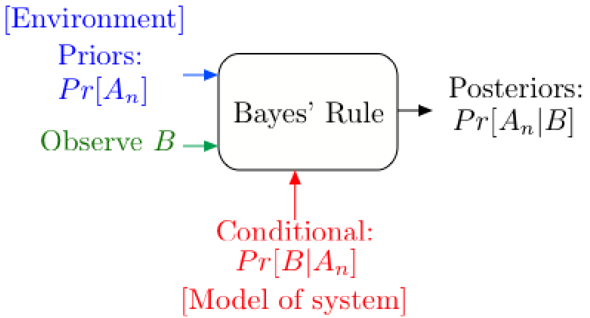

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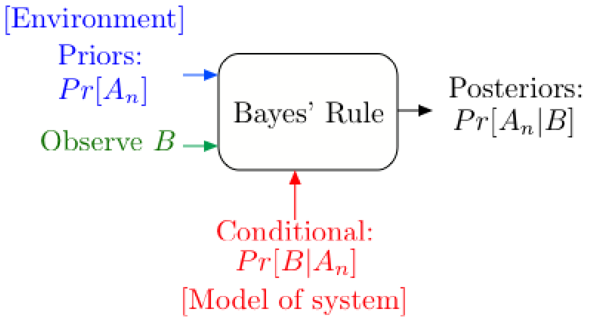

Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$Pr[B|A]$ →

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Random Experiment: Pick a random male.

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$$Pr[A|B]???$$

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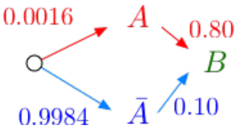

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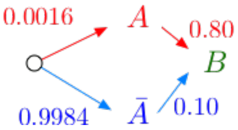

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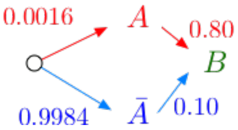

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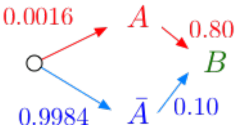

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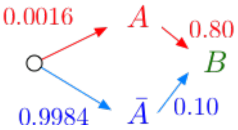

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A 1.3% chance of prostate cancer with a positive PSA test.

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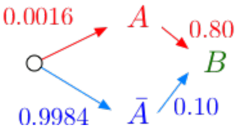

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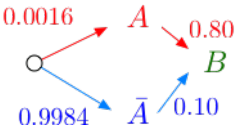

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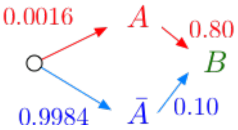

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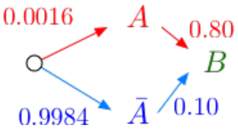

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$$P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = .013.$$

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...

Incontinence..

Death.

Quick Review

Events, Conditional Probability, Independence, Bayes' Rule

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  $$Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A].$$

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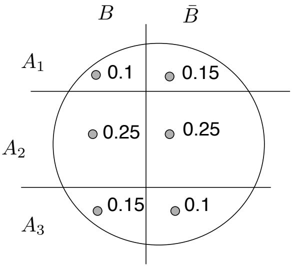

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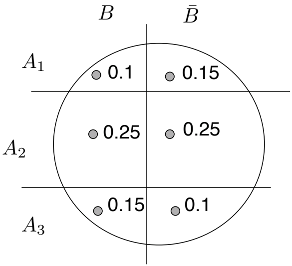

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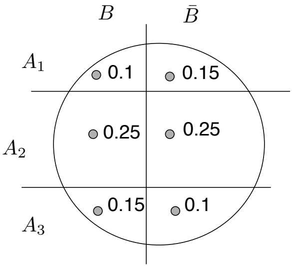

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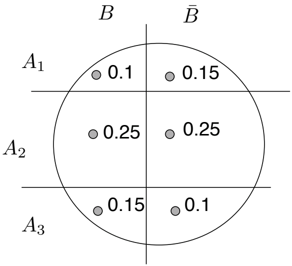

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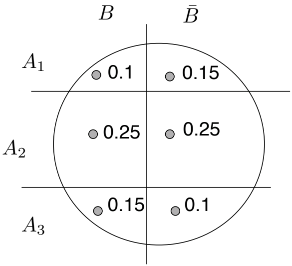

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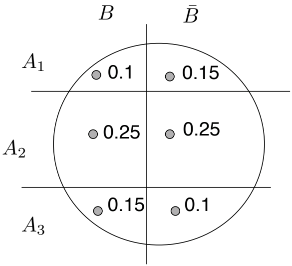

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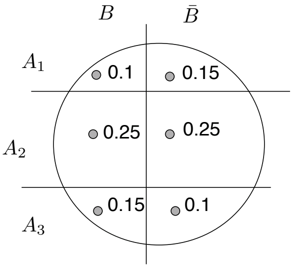

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Pairwise Independence

Flip two fair coins. Let

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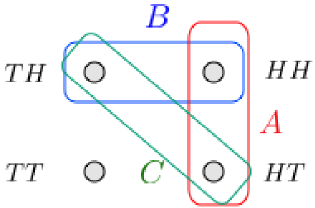

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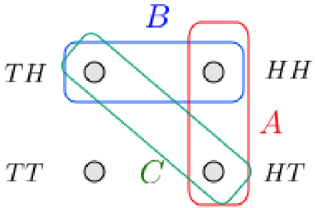

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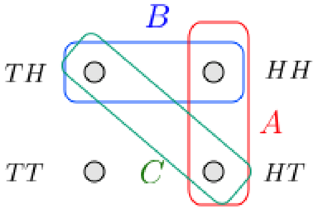

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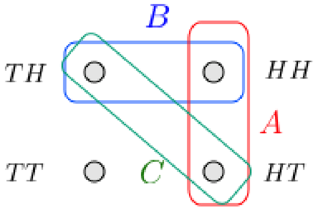

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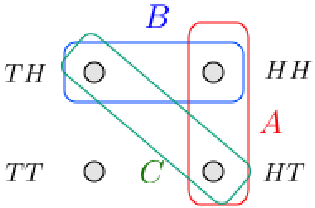

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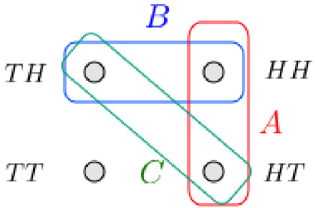

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If A did not say anything about C and B did not say anything about C, then $A \cap B$ would not say anything about C.

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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Proof:
See Notes 25, 2.7. □