How to read like a mathematician

Maxim #1 When you read mathematical writing and you come across a proof

Maxim #2 At the heart of most proofs, definitions, etc., there is a simple idea.
Induction
The sledgehammer of math

P(n): propositional function with domain N

I tell you:

Question

Principle of induction To prove

enough to prove:
Base Case:
Inductive Step:
Principle of induction: To prove $\forall n \in \mathbb{N} \ P(n)$, it is enough to prove:

**Base Case:** $P(0)$

**Inductive Step:** $\forall n \in \mathbb{N} \ (P(n) \Rightarrow P(n+1))$

**Question:** Would this work if we replaced $\mathbb{N}$ with $\mathbb{Z}$?
Examples of using induction

Question: What is $0 + 1 + 2 + \ldots + n$?

Claim: How to prove it?
Thm. For all $n \in \mathbb{N}$, $0 + 1 + \ldots + n = \frac{n(n+1)}{2}$

proof

Base Case:

Inductive hypothesis:

Inductive step:
2.1 Another example

Triomino tile:

Tiling a grid with triominos:

Question: Can you tile a chessboard with triominos?

Easier version:

Thm
For every \( n \in \mathbb{N} \), any \( 2^n \times 2^n \) grid with one square removed can be tiled with trominos.

**Proof**

**Base case:**

**Inductive hypothesis:** Assume

**Inductive step:** Have a \( 2^{n+1} \times 2^{n+1} \) grid with one square removed. Want to show
Question: Tile this grid with trinominoes. The induction argument tells you how!
③ When is induction useful?

When you want to prove a statement about all natural numbers
Or all natural numbers in some range

Example

Especially if

①

Example

②
Variations on induction

1. Different base case

2. Multiple base cases

3. Change the inductive assumption

4. Strong induction
4.1 Changing the inductive hypothesis

For every $n \in \mathbb{N}$,

- **Base case:**

- **Inductive hypothesis:**

- **Inductive step:**
Then for every $n \in \mathbb{N}$, the sum of the first $n$ odd numbers is a perfect square.

It would help to know what the sum of the first $n$ odd numbers is equal to.

proof: We will show

Base Case:

Inductive hypothesis:

Inductive step:
Regular induction

Strong induction
Then Every natural number

Example

proof attempt Base case:

Inductive hypothesis:

Inductive step:

Two cases
Case 1:
Case 2:
Then every natural number $n > 1$ is a product of prime numbers.

Example 12 = 2·2·3  17 = 17  57 = 3·19  60 = 15·4 = 5·3·2·2

Proof attempt

Base case: $n = 2$

\[ 2 \text{ is prime} \checkmark \]

Inductive hypothesis: Assume $n > 1$ and

Inductive step: WTS $n+1$ is a product of primes

Two cases

Case 1: $n+1$ is prime \checkmark
Case 2: $n+1$ is not prime

\[ \exists a, b, (n+1) = a·b \text{ and } a, b \neq 1, a, b \neq n+1 \]
Aside: Induction, Strong induction, well-ordering
Not very important to understand

Induction

Strong induction

Well-ordering

Strong induction on \( P(n) \) =

Induction on \( P(n) \) from well ordering:
Different base case & multiple base cases

Fibonacci sequence

How fast does the Fibonacci sequence grow?

<table>
<thead>
<tr>
<th>n</th>
<th>F_n</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It looks like

Ihm

How to prove it?
Then for all natural numbers $n \geq 6$, $F_n > n$

Proof: Base cases:

Inductive hypothesis:

Inductive step:

Exercise: Harder:
5 Be Careful

Lecture 2 notes:

Thus

proof Let P(n) =

Base case:
Inductive hypothesis:
Inductive step:
Thus, all people have the same number of hairs on their head (!?)

**Proof**

Let $P(n) =$ "in any set of $n$ people, everyone has the same number of hairs on their head."

Show $P(n)$ by induction

**Base case:** $n = 1$ 1 person $\checkmark$

**Inductive hypothesis:** Assume $P(n)$

**Inductive step:** $n+1$ people $a_1, a_2, \ldots, a_n, a_{n+1}$

$IH \Rightarrow a_1, a_2, \ldots, a_n$ have same # of hairs $a_2, a_3, \ldots, a_{n+1}$ have same # of hairs

So all have same # of hairs as $a_2$

What's wrong with this?