How to read like a mathematician

Maxim #1 When you read mathematical writing and you come across a proof, cover it up and try to figure it out yourself. You may not succeed, but you will understand what obstacles the proof has to overcome.

Maxim #2 At the heart of most proofs, definitions, etc., there is a simple idea. Your mission: find it!
**Induction**

The sledgehammer of math

\[ P(n) : \text{propositional function with domain } \mathbb{N} \]

\[ \text{I tell you: } P(0) \text{ is true} \]

\[ P(n) \Rightarrow P(n+1) \text{ for all } n \in \mathbb{N} \]

**Question** Is \( P(1000) \) true? **Yes!**

\[ \begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \ldots & \quad 999 & \quad 1000 \\
\end{align*} \]

**Principle of induction**

To prove \( \forall n \in \mathbb{N} \ P(n) \), it is enough to prove:

- **Base Case**: \( P(0) \)

- **Inductive Step**: \( \forall n \in \mathbb{N} \ (P(n) \Rightarrow P(n+1)) \)
Principle of induction

To prove \( \forall n \in \mathbb{N} \ P(n) \), it is enough to prove:

Base Case: \( P(0) \)

Inductive Step: \( \forall n \in \mathbb{N} \ (P(n) \Rightarrow P(n+1)) \)

Question: Would this work if we replaced \( \mathbb{N} \) with \( \mathbb{Z} \)? No!

Counterexample:

\[ P(n) = \text{“} n \geq -2 \text{”} \text{ with domain } \mathbb{Z} \]

\[ P(0) = \text{“} 0 \geq -2 \text{”} \checkmark \]

\[ P(n) \Rightarrow P(n+1) \text{ “} n \geq -2 \Rightarrow n+1 \geq -2 \text{”} \checkmark \]

\[ \ldots -4 -3 -2 -1 0 1 2 3 \ldots \]
Examples of using induction

**Question**  What is $0+1+2+\ldots+n$?

\[
\begin{align*}
0 & = \frac{0(0+1)}{2} = 0 \\
0+1 & = 1 = \frac{1(1+1)}{2} = 1 \\
0+1+2 & = 3 = \frac{2(2+1)}{2} = 3 \\
0+1+2+3 & = 6 = \frac{3(3+1)}{2} = 6
\end{align*}
\]

**Claim**  $0+1+2+\ldots+n = \frac{n(n+1)}{2}$

**How to prove it?**  Induction!
Thm. For all \( n \in \mathbb{N} \), \( 0 + 1 + ... + n = \frac{n(n+1)}{2} \)

proof. 

**Base Case:** \( n = 0 \). \( 0 = \frac{0(0+1)}{2} \)

**Inductive Hypothesis:** Assume \( 0 + 1 + ... + n = \frac{n(n+1)}{2} \)

**Inductive Step:** Use \( \text{IH} \) to show that

\[
0 + 1 + ... + n + (n+1) = \frac{n(n+1)}{2} + (n+1) \quad \text{(By IH)}
\]

\[
= \frac{n(n+1) + 2(n+1)}{2}
\]

\[
= \frac{n(n+1) + 2(n+1)}{2}
\]

\[
= \frac{(n+2)(n+1)}{2}
\]

\[
= \frac{(n+1)(n+1)}{2}.
\]
2.1 Another example

Triomino tile: 

Tiling a grid with triominoes:
cover all squares, no overlaps

Question: Can you tile a chessboard with triominoes?
No! Chessboard has $8 \times 8 = 64$ squares, not divisible by 3.

What if we remove one square?

Easier version: tile a $4 \times 4$ grid with 1 square removed?

Thm: For every $n \in \mathbb{N}$, any $2^n \times 2^n$ grid with one square removed can be tiled with triominoes.
Thus, for every $n \in \mathbb{N}$, any $2^n \times 2^n$ grid with one square removed can be tiled with trominos.

**Proof:**

**Base case:** $n=0$. $2^0 = 1$

- $1 \times 1$ grid
- 1 square removed
- Nothing left, easy to tile!

**Inductive hypothesis:** Assume all $2^n \times 2^n$ grids with one square removed can be tiled.

**Inductive step:** Have a $2^{n+1} \times 2^{n+1}$ grid with one square removed. Want to show it can be tiled.

1. Divide into 4 $2^n \times 2^n$ grids
2. Place tromino in center
3. Now each $2^n \times 2^n$ piece is missing one piece
4. Tile each one using IH
Question: Tile this grid with triominoes

The induction argument tells you how!

Answer:

First, divide into 4 $2 \times 2$ grids & put a triomino in the center.

Now, recursively tile each $2 \times 2$ grid.

Lesson: proofs by induction often tell you recursive algorithms.
③ When is induction useful?

When you want to prove a statement about all natural numbers
Or all natural numbers in some range

Example \( \forall n \in \mathbb{N} \ (0+1+2+...+n = \frac{n(n+1)}{2}) \)

Especially if

① You are proving something about a recursively defined sequence
Example Fibonacci numbers
\( F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \)
0, 1, 1, 2, 3, 5, 8, ...

② You are proving something about a recursive algorithm or an algorithm that loops

cs 170!
Variations on induction

1. **Different base case**
   - Prove $P(7)$ and $P(n) \Rightarrow P(n+1)$
   - Conclude $\forall n \geq 7 \ P(n) = \forall n \in \mathbb{N} \ (n \geq 7 \Rightarrow P(n))$

2. **Multiple base cases**
   - Prove $P(0)$, $P(1)$, $P(2)$ and $\forall n \geq 2 \ (P(n) \Rightarrow P(n+1))$
   - Conclude $\forall n \in \mathbb{N} \ P(n)$

3. **Change the inductive assumption**
   - Want to prove $\forall n \in \mathbb{N} \ P(n)$
   - Find some $Q(n)$ such that $Q(n) \Rightarrow P(n)$
   - Prove $Q(0)$ and $Q(n) \Rightarrow Q(n+1)$
   - Looks harder, but useful if $P(n) \Rightarrow P(n+1)$ hard to prove because $P$ is too weak

4. **Strong induction**
   - Prove $P(0)$ and $(P(0) \land P(1) \land \ldots \land P(n)) \Rightarrow P(n+1)$
   - Conclude $\forall n \in \mathbb{N} \ P(n)$
   - Secretly just a special case of 3
4.1 Changing the inductive hypothesis

Theorem: For every \( n \in \mathbb{N} \), the sum of the first \( n \) odd numbers is a perfect square. i.e., \( = k^2 \) for some \( k \in \mathbb{N} \)

Proof attempt

Base case: \( n = 1 \)
- First odd number: 1
- \( 1 = 1^2 \) ✓

Inductive hypothesis: Sum of first \( n \) odd numbers is perfect square

Want to show

Inductive step: WTS: Sum of first \( n+1 \) odd numbers is perfect square
- \((n+1)^{st}\) odd number: \( 2n+1 \)

\[
\begin{align*}
\text{sum of first } n+1 \text{ odd } #s &= (\text{sum of first } n \text{ odd } #s) + (2n+1) \\
&= k^2 + (2n+1) \text{ for some } k \in \mathbb{N} \\
&= ?? \checkmark
\end{align*}
\]

It would help to know what the sum of the first \( n \) odd numbers is equal to
Then for every $n \in \mathbb{N}$, the sum of the first $n$ odd numbers is a perfect square.

It would help to know what the sum of the first $n$ odd numbers is equal to:

- $1 = 1 = 1^2$
- $1 + 3 + 5 + 7 = 16 = 4^2$
- $1 + 3 = 4 = 2^2$
- $1 + 3 + 5 = 9 = 3^2$

It looks like the sum of first $n$ odd numbers is $n^2$.

**Proof** We will show the sum of the first $n$ odd numbers is $n^2$.

- **Base Case:** $n = 1$
  
  $1 = 1^2$

  (Actually, base case of $n = 0$ also works.)

- **Inductive Hypothesis:** Sum of first $n$ odd numbers is $n^2$.

- **Inductive Step:** WTS sum of first $(n+1)$ odd numbers is $(n+1)^2$.

  Sum of first $(n+1)$ odd numbers = (Sum of first $n$) + $(2n+1)$

  $= n^2 + (2n+1)$ (By IH)

  $= (n+1)^2$

**Lesson:** When you get stuck, work out small examples.
4.2 Strong induction

Regular induction

Prove $P(0)$
Prove $P(n) \Rightarrow P(n+1)$
Conclude $\forall n \in \mathbb{N}, P(n)$

Strong induction

Prove $P(0)$
Prove $(\forall k \leq n, P(k)) \Rightarrow P(n+1)$
Conclude $\forall n \in \mathbb{N}, P(n)$
Then Every natural number $n > 1$ is a product of prime numbers

Example $12 = 2 \cdot 2 \cdot 3$ $17 = 17$ $57 = 3 \cdot 19$ $60 = 15 \cdot 4 = 5 \cdot 3 \cdot 2$ $2$

proof attempt

Base case: $n = 2$

2 is prime ✓

Inductive hypothesis: Assume $n > 1$ and $n$ is a product of prime numbers ← need to replace this

Inductive step: WTS $n+1$ is a product of primes

Two cases

Case 1: $n+1$ is prime ✓

Case 2: $n+1$ is not prime

$\exists a, b, (n+1) = a \cdot b$ and $a, b \neq 1$, $a, b \neq n+1$

How can we use IH??

Done if we had IH for $a$ and $b$!
Then every natural number \( n > 1 \) is a product of prime numbers.

Example: \( 12 = 2 \cdot 2 \cdot 3 \), \( 17 = 17 \), \( 57 = 3 \cdot 19 \), \( 60 = 15 \cdot 4 = 5 \cdot 3 \cdot 2 \cdot 2 \)

Proof attempt:

Base case: \( n = 2 \) (Base case of \( n = 0 \) actually works)

\( n = 2 \) is prime \( \checkmark \)

Inductive hypothesis: Assume \( n > 1 \) and for all \( k \leq n \), if \( k > 1 \) then \( k \) is a product of prime numbers.

Inductive step: WTS \( n+1 \) is a product of primes.

Two cases:

Case 1: \( n+1 \) is prime \( \checkmark \)

Case 2: \( n+1 \) is not prime

\( \exists a, b \), \( (n+1) = a \cdot b \) and \( a, b \neq 1 \), \( a, b \neq n+1 \)

\( 1 < a/b \leq n \) \( \Rightarrow \) \( a \) and \( b \) both product of primes

\( \Rightarrow \) \( n+1 \) is a product of primes.
4.3 Aside: Induction, Strong induction, well-ordering

Not very important to understand

Induction \[ P(0) \land P(n) \Rightarrow P(n+1) \quad \forall n \in \mathbb{N}, P(n) \]

Strong induction \[ P(0) \land \left( \forall k \leq n, P(k) \right) \Rightarrow P(n+1) \quad \forall n \in \mathbb{N}, P(n) \]

Well-ordering All nonempty subsets of \( \mathbb{N} \) have a least element

They are all equivalent

Strong induction on \( P(n) \) = Regular induction on \( Q(n) = \forall k \leq n, P(k) \)

Induction on \( P(n) \) from well ordering: Look at set of places where \( P(n) \) doesn't hold
Different base case & multiple base cases

Fibonacci sequence  \( F_0 = 0 \quad F_1 = 1 \quad F_{n+2} = F_n + F_{n+1} \)

\[
\begin{array}{c|c|c|c}
F_0 &=& 0 \\
F_1 &=& 1 \\
F_2 &=& 0+1=1 \\
F_3 &=& 1+1=2 \\
F_4 &=& 1+2=3 \\
F_5 &=& 2+3=5 \\
F_6 &=& 3+5=8 \\
\end{array}
\]

How fast does the Fibonacci sequence grow?
Linear? Quadratic? Exponential? …

<table>
<thead>
<tr>
<th>n</th>
<th>0 1 2 3 4 5 6 7 8 9 …</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_n )</td>
<td>0 1 1 2 3 5 8 13 21 34 …</td>
</tr>
</tbody>
</table>

It looks like eventually \( F_n \) is always bigger than \( n \)

**Theorem** For all \( n \geq 6 \), \( F_n > n \).

How to prove it? **Induction**!
Then For all natural numbers $n \geq 6$, $F_n > n$

**proof**  

**Base cases:**  

$n = 6$ \quad $F_6 = 8 > 6$  

$n = 7$ \quad $F_7 = 13 > 7$

**Inductive hypothesis:** Assume $n \geq 7$ and for all $k \leq n$, if $k \geq 6$ then $F_k > k$  

**Inductive step:** WTS $F_{n+1} > n+1$

$$F_{n+1} = F_n + F_{n-1}$$

$$> (n-1) + n \quad \text{By IH, note } n \geq 7 \Rightarrow n,n-1 \geq 6$$

$$= n+1 + (n-2)$$

$$> n+1 \quad n \geq 7 \Rightarrow n-2 > 0$$

**Exercise:** $F_n < 2^n$ for all $n$

**Harder:** Find an exact formula for $F_n$
Lecture 2 notes: There are at least 2 people in San Francisco who have the same # of hairs on their head.

Thus, all people have the same number of hairs on their head (?!)

Proof: Let \( P(n) = \) "in any set of \( n \) people, everyone has the same number of hairs on their head.

Show \( P(n) \) by induction.

Base case: \( n = 1 \) 1 person \( \checkmark \)

Inductive hypothesis: Assume \( P(n) \)

Inductive step: \( n+1 \) people \( a_1, a_2, \ldots, a_n, a_{n+1} \)

\( \exists H \Rightarrow a_1, a_2, \ldots, a_n \) have same # of hairs

\( a_2, a_3, \ldots, a_{n+1} \) have same # of hairs

So, all have same # of hairs as \( a_2 \)

What's wrong with this?
Thus all people have the same number of hairs on their head (?!)

**Proof**

Let \( P(n) \) = "in any set of \( n \) people, everyone has the same number of hairs on their head."

Show \( P(n) \) by induction.

**Base case:** \( n = 1 \) → 1 person → \( \checkmark \)

**Inductive hypothesis:** Assume \( P(n) \)

**Inductive step:** \( n+1 \) people \( a_1, a_2, \ldots, a_n, a_{n+1} \)

\[ \text{IH } \Rightarrow \quad a_1, a_2, \ldots, a_n \text{ have same # of hairs} \]

\[ a_2, a_3, \ldots, a_{n+1} \text{ have same # of hairs} \]

So all have same # of hairs as \( a_2 \)

What's wrong with this?

**Answer:** Inductive step doesn't work when \( n = 1 \)