Some advice on solving problems

Maxim #1  Don't give up after 30 seconds!
Common pattern: look at a problem, can't think of anything right away, give up.
Instead: Think about the problem for 5-10 minutes, often you will at least come up with some things to try.

Maxim #2  Do the messy calculations/hard work!
Common pattern: you have an idea for how to solve a problem but it looks hard to do & maybe won't work anyway, so you look for an easier approach.
Instead: Just try it! 30% of the time it will work.
0 Introduction

Question: Are there more real numbers than natural numbers?

?? There are infinitely many of both! What does “more” mean?

Goal for today: Turn this into a mathematical question and answer it.

Key step: finding a reasonable mathematical definition of when two infinite sets have “the same number” of elements.

Not so important for CS. But an excuse to introduce useful mathematical terminology. Plus it’s super cool!
② Sets

Def A set is an unordered collection of objects. Two sets are equal if they contain the same objects.

Example \( \{1,2,3\} = \{3,1,2\} \) \( \leftarrow \) bag
\( \{2,3,5\} = \text{set of all prime numbers} < 6 \)
\( \{1,2,3\} \neq \{1,2,3,4\} \)

Def \( x \in A \) means \( x \) is an element of \( A \)
\( x \notin A \) means \( x \) is not an element of \( A \)

Example \( 2 \in \{1,2,3\} \), \( 0 \notin \{1,2,3\} \), \( \sqrt{2} \in \mathbb{R}, \sqrt{2} \notin \mathbb{Q} \)

Def A set \( A \) is a subset of \( B \), written \( A \subseteq B \), if for every \( x \in A \) we have \( x \in B \)

Example \( \{1,2,3\} \subseteq \{1,2,3,4\} \), \( \mathbb{N} \subseteq \mathbb{R}, \{1,2\} \subseteq \{1,2\} \), \( \{1\} \notin \mathbb{N} \)
Question: Suppose \( A \subseteq B \) and \( B \subseteq A \). Must \( A = B \)? Yes.

How to prove two sets are equal: \( \forall x \ (x \in A \iff x \in B) \)

1. Let \( x \) be an arbitrary element of \( A \).
   Show \( x \in B \).

2. Let \( x \) be an arbitrary element of \( B \).
   Show \( x \in A \).
2. Common sets

\[ \mathbb{N} \text{ natural numbers } \{0, 1, 2, \ldots\} \]

\[ \mathbb{Z} \text{ integers } \{-\ldots, -2, -1, 0, 1, 2, \ldots\} \]

\[ \mathbb{Q} \text{ rational numbers } \frac{1}{2} \in \mathbb{Q}, \pi \notin \mathbb{Q} \]

\[ \mathbb{R} \text{ real numbers } \sqrt{2} \in \mathbb{R}, \pi \in \mathbb{R} \]

\[ \emptyset \text{ empty set nothing is an element of } \emptyset \]

Question ① \( 3 \in \mathbb{R} \)? Yes ④ \( \emptyset \subseteq \mathbb{N} \)? Yes (vacuously)

② \( 3 \in \emptyset \)? No! ⑤ \( \mathbb{N} \subseteq \emptyset \)? No!

③ \( \mathbb{N} \subseteq \mathbb{R} \)? No ⑥ Steph Curry \( \in \emptyset \)? No!

\( \text{not a real number!} \)
2.2 How to describe a set

① List the elements \{18, 26, -3\}, \{Khalil, Shahzar, Patrick\}

② List enough elements to make it clear what set it is
\{0, 2, 4, 6, ...\} even natural numbers
\{1, 2, 3, ..., 10\} natural numbers between 1 and 10
Bad: \{1, ...\} all numbers \(\geq 1\)? all odd numbers?

③ Describe it in words “the set of all odd natural numbers”

④ Set builder notation \(\{x \in A \mid P(x)\}\)
(a.k.a. set comprehension) I can just write \(\{x \mid P(x)\}\)

\(\{n \in \mathbb{N} \mid \exists m \in \mathbb{N} (n = 2 \cdot m)\}\) = \{0, 2, 4, 6, ...\}
\(\{x \in \mathbb{R} \mid x^2 = -x\}\) = \{0, -1\}
\(\{x \in \mathbb{N} \mid x^2 = -x\}\) = \{0\}

⑤ Some specialized notation. \(a \leq b\) real numbers
\([a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}\)
### 2.3 Operations on sets

1. **Union**
   \[ A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \]
   
   \[ \{1,2\} \cup \{1,7\} = \{1,2,7\} \]
   
   \[ \{1,2,3\} \cup \emptyset = \{1,2,3\} \]

2. **Intersection**
   \[ A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \]
   
   \[ \{1,2\} \cap \{1,7\} = \{1\} \]
   
   \[ \{1,2,3\} \cap \{4,3\} = \emptyset \]

3. **Difference**
   \[ A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \} \]
   
   \[ \{1,2,3\} \setminus \{1,7\} = \{2\} \]

4. **Complement**
   Assume \( A \subseteq U \).
   \[ A^c = U \setminus A \]
   
   \[ \mathbb{Q}^c = \{ \text{irrational numbers} \} \]
   
   \( \mathbb{R} \) is implicit.
5. Cartesian product $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

$\{1, 2\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$

Also $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \forall i \}$

Example $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \leftarrow EE16A/\text{Math 54}$

6. Powerset $\mathcal{P}(A) = \text{set of all subsets of } A$

$\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Question $\emptyset \in \mathcal{P}(A)$ for any $A$? Yes. $\emptyset \subseteq A$

$\mathbb{N} \in \mathcal{P}(\mathbb{R})$? Yes. $\mathbb{N} \subseteq \mathbb{R}$

$A \subseteq B \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$? Yes. $x \subseteq A \Rightarrow x \subseteq B$
2.4 Cardinality of finite sets

Def If A is a finite set, write $|A|$ to denote the number of elements in A.

$|\{-1, 7\}| = 2$

$|\{Khalil, Shahzar, Patrick\}| = 3$

Question If $|A|=18=2$, what is $|A \times B|$?

$A = \{a_1, a_2\}$ \hspace{1cm} $B = \{b_1, b_2\}$

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$

Exercise If $|A|=n$ and $|B|=k$ then $|A \times B|=nk$

If $|A|=n$ then $|P(A)|=2^n$
### Functions

**Def.** If $A$ and $B$ are sets, a **function** $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$.

**Example** $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$

- $f(1) = a$
- $f(2) = b$
- $f(3) = a$

**Def.** The **range** of a function $f : A \rightarrow B$, written $\text{range}(f)$, is the set $\{b \in B \mid \exists a \in A \ (f(a) = b)\}$.

**Example** With $f$ and $g$ as above,

- $\text{range}(f) = \{a, b\}$
- $\text{range}(g) =$ set of all real numbers $\geq 0$. 

$g : \mathbb{R} \rightarrow \mathbb{R}$

$g(x) = x^2$
(3.1) Anatomy of a function

\[ f : A \rightarrow B \]

- **domain**
- **codomain** (part of the def. of \( f \))

\[ f(1) = a \]

\( \text{Image of } 1 \text{ under } f \)
3.2 How to describe functions

1. Explicitly list where each element of the domain is sent
   \[ f: \{ \text{Khalil, Shahzar, Patrick} \} \rightarrow \{1, 2, 3\} \]
   \[ f(\text{Khalil}) = 1 \quad f(\text{Shahzar}) = 1 \quad f(\text{Patrick}) = 1 \]

2. Write down a formula that says where each element of the domain is sent
   \[ g: \mathbb{N} \rightarrow \mathbb{N} \quad g(x) = x + 1 \]
   Bad \[ h: \mathbb{N} \rightarrow \mathbb{N} \quad h(x) = x - 1 \quad h(0) \notin \mathbb{N} \quad \mathbb{N} \rightarrow \mathbb{Z} ? \]
   Bad \[ k: \mathbb{R} \rightarrow \mathbb{R} \quad k(x) = \frac{1}{x} \quad k(0) = \text{?} ? \quad \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} ? \]

3. Describe in words where each elt of the domain is sent
   \[ j: \{0, 1, 2, \ldots, 10\} \rightarrow \mathbb{N} \]
   \[ j(n) = \text{the smallest natural number with at least } n \text{ letters when written in English} \]
   \[ j(4) = 0 \quad j(5) = 3 \]

4. Definition by cases
   \[ l: \mathbb{N} \rightarrow \mathbb{N} \]
   \[ l(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n-\frac{1}{2} & \text{if } n \text{ is odd} \end{cases} \]
   \[ l(2) = 1 \quad l(9) = 4 \]
3.3 Operations on Functions

Def If $f : A \to B$ and $g : B \to C$ are functions, the **composition** of $g$ and $f$, denoted $g \circ f$, is the function from $A$ to $C$ defined by

$$(g \circ f)(a) = g(f(a))$$

"first do $f$ then do $g"$

Example

$f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

$g : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$

$(g \circ f) : \mathbb{N} \to \mathbb{R}$

$f(n) = (n, n+1)$

$g((n, m)) = \sqrt{n \cdot m}$

$(g \circ f)(n) = g((n, n+1)) = \sqrt{n(n+1)}$
Have a function \( f : A \to B \)

**Def:** If \( C \subseteq A \), the **image of** \( C \) **under** \( f \), written \( f(C) \), is the set \( \{ b \in B \mid \exists x \in C \ (f(x) = b) \} \)

\[
\begin{array}{c}
\text{C} \\
\text{A} \\
\end{array}
\begin{array}{c}
\text{f(C)} \\
\text{B} \\
\end{array}
\]

\( f(C) \subseteq B \)

**Def:** If \( D \subseteq B \), the **inverse image** of \( D \) **under** \( f \), written \( f^{-1}(D) \), is the set \( \{ a \in A \mid f(a) \in D \} \)

\[
\begin{array}{c}
\text{f}^{-1}(D) \\
\text{A} \\
\end{array}
\begin{array}{c}
\text{D} \\
\text{B} \\
\end{array}
\]

"things that get sent into \( D \)"

\( f^{-1}(D) \subseteq A \)
4) Special Properties of Functions

Def: A function $f: A \rightarrow B$ is **injective** if for all $a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

Def: A function $f: A \rightarrow B$ is **surjective** if for all $b \in B$ there is some $a \in A$ such that $f(a) = b \Rightarrow \text{range}(f) = B$
Tip: To prove $f$ injective, use contrapositive

\[ \forall a_1, a_2 \in A \ (a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)) \]
\[ \equiv \forall a_1, a_2 \in A \ (f(a_1) = f(a_2) \Rightarrow a_1 = a_2) \]

Examples

① $f : \mathbb{N} \to \mathbb{N}$

- injective: $2n = 2m \Rightarrow n = m$
- not surjective: $1 \notin \text{range}(f)$

- $f(n) = 2n$

② $g : \mathbb{Z} \to \mathbb{N}$

- not injective: $|1 - 2| = |2 - 1|
- surjective: if $n \in \mathbb{N}$ then $g(n) = n$

- $g(n) = \lfloor n \rfloor$

③ $h : \mathbb{Z} \to \mathbb{Z}$

- injective: $n + 1 = m + 1 \Rightarrow n = m$
- surjective: $\forall n, h(n - 1) = n$

- $h(n) = n + 1$

④ $K : \{1, 2\} \to \{a, b, c\}$

- not injective: $K(1) = K(2)$
- not surjective: $b \notin \text{range}(K)$

- $K(1) = a$
- $K(2) = a$
Def: A function $f: A \rightarrow B$ is bijective if it is both injective and surjective. $f$ is called a bijection.

If $f: A \rightarrow B$ is a bijection, the inverse of $f$, denoted $f^{-1}$, is the function from $B$ to $A$ defined by $f^{-1}(b) =$ the unique $a$ such that $f(a) = b$.

For all $a \in A$, $f^{-1}(f(a)) = a$

For all $b \in B$, $f(f^{-1}(b)) = b$
5. **Cardinality**

Are there the same number of △ and □?

Yes! 4 of each

Are there the same number of ○ and *?

Yes!

Did you have to count? No!

Can match up ○ and * so that all are paired up

I.e. there is a bijection between the two sets!

Idea for today: Extend this definition to infinite sets!
An aside

Ask a very young child:

Are there the same number of △ and □?

△ △ △ △ △ △ they will say “yes”

□ □ □ □ □ □

Are there the same number of ○ and ★?

○ ○ ○ ○ ○ ○ they will say “no”

★ ★ ★ ★ ★ ★

Counting is not that obvious! But bijections are?
5.1 Definition of Cardinality

**Definition** Two sets $A$ and $B$ have the **same** cardinality, written $|A| = |B|$, if there is a bijection $f: A \rightarrow B$

**Example** $|\{1, 2, 3\}| = |\{a, b, c\}|$ $f: \{1, 2, 3\} \rightarrow \{a, b, c\}$

$f(1) = a$  $f(2) = b$  $f(3) = c$

**Exercise**

① Show $|A| = |B| \Rightarrow |B| = |A|$

*Hint:* inverse of a bijection

② Show $(|A| = |B| \land |B| = |C|) \Rightarrow |A| = |C|$

*Hint:* composition of functions
**Theorem** Let $A$ be the set of even natural numbers. Then $|\mathbb{N}| = |A|$.  

**Proof**

$0, 2, 4, 6, 8, \ldots$

$f: \mathbb{N} \rightarrow A$ defined by $f(n) = 2n$ is a bijection.

**Theorem** $|\mathbb{N}| = |\mathbb{Z}|$

**Proof**

$0 \rightarrow 0$

$f: \mathbb{N} \rightarrow \mathbb{Z}$ defined by

$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ even} \\ \frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$
positive rational numbers

Then \( |\mathbb{N}| = |\mathbb{Q}^+| \)

**proof**

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<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</thead>
<tbody>
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<td>1</td>
<td>( \frac{1}{1} \rightarrow \frac{1}{2} \rightarrow \frac{1}{3} \rightarrow \frac{1}{4} \rightarrow \frac{1}{5} )</td>
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<tr>
<td>2</td>
<td>( \frac{2}{1} \rightarrow \frac{2}{2} \rightarrow \frac{2}{3} \rightarrow \frac{2}{4} \rightarrow \frac{2}{5} )</td>
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<tr>
<td>3</td>
<td>( \frac{3}{1} \rightarrow \frac{3}{2} \rightarrow \frac{3}{3} )</td>
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<tr>
<td>4</td>
<td>( \frac{4}{1} \rightarrow \frac{4}{2} )</td>
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</tbody>
</table>

1. Make infinite grid with \( \mathbb{N}^+ \) on each axis
2. Follow zig-zag path
3. Cross out duplicates

**Exercise**

1. \( |\mathbb{N}| = |\mathbb{N} \times \mathbb{N}| \)
2. \( |\mathbb{N}| = |\mathbb{Q}| \)
5.2 More about cardinality

**Def** If $A$ and $B$ are sets, we say the cardinality of $A$ is less than or equal to the cardinality of $B$, written $|A| \leq |B|$, if there is an injective function $f : A \to B$

**Example** $|\{1, 2, 3\}| \leq |\{a, b, c, d, e\}|$

**Note:** $|A| = |B| \implies |A| \leq |B|$ A bijection is an injection

**Theorem (Cantor-Schroeder-Bernstein)** If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. \(\rightarrow\) Not that easy!

**Meaning:** to show $|A| = |B|$, enough to find injections going both ways
(6) Countability

Def A set $A$ is **countable** if $|A| \leq |\mathbb{N}|$.

Some countable sets

- Any finite set
- Set of even natural numbers $\mathbb{N}$, $2\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}$
- Set of all finite binary sequences $\{0, 1, 00, 01, 10, 11, 000, \ldots \}$
- Set of all valid Java programs

A set is countable if it is either finite or has the same cardinality as $\mathbb{N}$.

$A$ is countable $=$ you can list the elements of $A$

i.e. $A = \{a_1, a_2, a_3, \ldots \}$
7. Uncountability

Def: A set is **uncountable** if it is not countable.

Are there any uncountable sets? **Yes!**

Thm (Cantor) $|\mathbb{N}| \neq |\mathbb{R}|$ (will prove on next slide)

Cor: $\mathbb{R}$ is not countable

$|\mathbb{N}| \leq |\mathbb{R}|$ \( f: \mathbb{N} \to \mathbb{R} \) \( f(n) = n \)

So if $|\mathbb{R}| \leq |\mathbb{N}|$ then by CSB, $|\mathbb{N}| = |\mathbb{R}|$, contradicting Cantor's theorem

Note: Bijection $f: \mathbb{N} \to \mathbb{R}$ = way to list all real numbers

\( r_0 = f(0), \ r_1 = f(1), \ r_2 = f(2), \ldots \)

\( \mathbb{R} = \{ r_0, \ r_1, \ r_2, \ldots \} \)
Thm (Cantor) $|\mathbb{N}| \neq |\mathbb{R}|$

Proof: Suppose we can list all real numbers:

$$R = \{r_0, r_1, r_2, r_3, \ldots \}$$

We will show there is a real number not on this list!

Write $r_i$'s in decimal notation

E.g. $r_0 = -17.0287651\ldots$

Comment: need to be a little careful

Define $r$ so that its $n$th digit after the decimal point is not equal to the $n$th digit of $r_n$

$r = 0.\, d_0 d_1 d_2 \ldots$ where $d_n = \begin{cases} \text{1} & \text{if } n \text{th digit of } r_n \text{ is 1} \\ 0 & \text{otherwise} \end{cases}$

$r$ disagrees with every $r_n$, so it is not on our list!