## Review.


Theory: If you drink you must be at least 18.

Which cards do you turn over?

Drink $\implies$ "$\geq 18$"

"$< 18$" $\implies$ Don't Drink.

## CS70: Lecture 2. Outline.

Today: Proofs!!!

1. By Example.
2. Direct. (Prove $P \implies Q$. )
3. by Contraposition (Prove $P \implies Q$)
4. by Contradiction (Prove $P$.)
5. by Cases

If time: discuss induction.

## Quick Background and Notation.

Integers closed under addition.

$a,b \in Z \implies a+b \in Z$

$a|b$ means "a divides b".

$2|4$? Yes! Since for $q=2$, $4=(2)2$.

$7|23$? No! No $q$ where true.

$4|2$? No!

Formally: $a|b \iff \exists q \in Z$ where $b = aq$.

$3|15$ since for $q=5$, $15=3(5)$.

A natural number $p>1$, is **prime** if it is divisible only by 1 and itself.

## Direct Proof.

**Theorem:** For any $a,b,c \in Z$, if $a|b$ and $a|c$ then $a|(b-c)$.

**Proof:** Assume $a|b$ and $a|c$
$b = aq$ and $c = aq'$ where $q,q' \in Z$

$b-c = aq - aq' = a(q-q')$ Done?

$(b-c) = a(q-q')$ and $(q-q')$ is an integer so

$a|(b-c)$ □

Works for $\forall a,b,c$?
Argument applies to *every* $a,b,c \in Z$.

Direct Proof Form:
Goal: $P \implies Q$
Assume $P$.
...
Therefore Q.

## Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$

Examples:
$n=121$ Alt Sum: $1-2+1=0$. Divis. by 11. As is 121.

$n=605$ Alt Sum: $6-0+5=11$ Divis. by 11. As is $605 = 11(55)$

**Proof:** For $n \in D_3$, $n = 100a + 10b + c$, for some $a,b,c$.

Assume: Alt. sum: $a-b+c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$

Left hand side is $n$, $k+9a+b$ is integer. $\implies 11|n$. □

Direct proof of $P \implies Q$:
Assumed $P$: $11|a-b+c$ . Proved $Q$: $11|n$.

## The Converse

Thm: $\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$

Is converse a theorem?
$\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Yes? No?

## Another Direct Proof.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$
**Proof:** Assume $11|n$.

$n = 100a + 10b + c = 11k \implies$
$99a + 11b + (a - b + c) = 11k \implies$
$a - b + c = 11k - 99a - 11b \implies$
$a - b + c = 11(k - 9a - b) \implies$
$a - b + c = 11\ell$ where $\ell = (k - 9a - b) \in Z$

That is $11|$alternating sum of digits. □

Note: similar proof to other. In this case every $\implies$ is $\iff$

Often works with arithmetic properties ...
...not when multiplying by 0.

We have.

Theorem: $\forall n \in N', (11|\text{alt. sum of digits of } n) \iff (11|n)$

## Proof by Contraposition

Thm: For $n \in Z^+$ and $d|n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do? Is it even true?
Hey, that rhymes ...and there is a pun ... colored blue.
Anyway, what to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$
...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$.

**Proof:** Assume $\neg Q$: $d$ is even. $d = 2k$.

$d|n$ so we have

$n = qd = q(2k) = 2(kq)$

$n$ is even. $\neg P$ □

## Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies$ $n$ is even. ($P \implies Q$)

$n^2$ is even, $n^2 = 2k$, ...$\sqrt{2k}$ even?

**Proof by contraposition:** $(P \implies Q) \equiv (\neg Q \implies \neg P)$

$P$ = '$n^2$ is even.' ........... $\neg P$ = '$n^2$ is odd'

$Q$ = 'n is even' ........... $\neg Q$ = 'n is odd'

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies$ $n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$

$n^2 = 2l + 1$ where $l$ is a natural number..

... and $n^2$ is odd!

$\neg Q \implies \neg P$ so $P \implies Q$ and ... □

## Proof by contradiction:form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in Z$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always "not" hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$

$\neg P \implies R \wedge \neg R \equiv$ False

or $\neg P \implies$ *False*

Contrapositive of $\neg P \implies$ *False* is *True* $\implies P$.
Theorem $P$ is proven. □

## Contradiction

**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in Z$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$

$$2b^2 = a^2 = 4k^2$$

$a^2$ is even $\implies$ $a$ is even.

$a = 2k$ for some integer $k$

$$b^2 = 2k^2$$

$b^2$ is even $\implies$ $b$ is even.
$a$ and $b$ have a common factor. Contradiction. □

## Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider number

$$q = (p_1 \times p_2 \times \cdots p_k) + 1.$$

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R ) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $x - q$,
- $\implies p | x - q \implies p \leq x - q = 1$.
- so $p \leq 1$. (**Contradicts** $R$.)

The original assumption that "the theorem is false" is false,
thus the theorem is proven. □

## Product of first $k$ primes..

Did we prove?

- "The product of the first $k$ primes plus 1 is prime."
- No.
- The chain of reasoning started with a false statement.

Consider example..

- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
- There is a prime *in between* 13 and $q = 30031$ that divides $q$.
- Proof assumed no primes *in between* $p_k$ and $q$.

## Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.
**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in Z$, *then both a and b are even.*

Reduced form $\frac{a}{b}$: $a$ and $b$ can't both be even! + Lemma
$\implies$ no rational solution. $\square$

**Proof of lemma:** Assume a solution of the form $a/b$.

$$\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0$$

Multiply by $b^5$,

$$a^5 - ab^4 + b^5 = 0$$

Case 1: $a$ odd, $b$ odd: odd - odd +odd = even. Not possible.
Case 2: $a$ even, $b$ odd: even - even +odd = even. Not possible.
Case 3: $a$ odd, $b$ even: odd - even +even = even. Not possible.
Case 4: $a$ even, $b$ even: even - even +even = even. Possible.

The fourth case is the only one possible, so the lemma follows. $\square$

## Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.
- $$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} * \sqrt{2}} = \sqrt{2}^2 = 2.$$

Thus, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).

One of the cases is true so theorem holds. $\square$

Question: Which case holds? Don't know!!!

## Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get
$4 = 3$.

By commutativity theorem holds. $\square$

Don't assume what you want to prove!

## Be really careful!

**Theorem:** $1 = 2$
**Proof:** For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$
$$x(x - y) = (x + y)(x - y)$$
$$x = (x + y)$$
$$x = 2x$$
$$1 = 2$$

$\square$

Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

$P \implies Q$ does not mean $Q \implies P$.

## Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False .

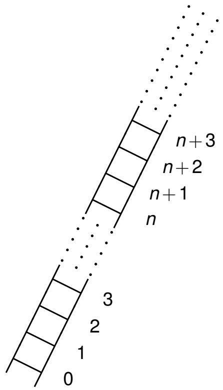
By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Either $\sqrt{2}$ and $\sqrt{2}$ worked.
or $\sqrt{2}$ and $\sqrt{2}^{\sqrt{2}}$ worked.

Careful when proving!
Don't assume the theorem. Divide by zero.Watch converse. ...

## CS70: Note 3. Induction!

1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.

## The natural numbers.




0, 1, 2, 3,
$\ldots, n, n+1, n+2, n+3, \ldots$

## A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

$\forall(n \in N) : P(n)$.

$P(n)$ is "$\sum_{i=0}^{n} i \frac{(n)(n+1)}{2}$".

Principle of Induction:

- Prove $P(0)$.
- Assume $P(k)$, "Induction Hypothesis"
- Prove $P(k+1)$. "Induction Step."

## Gauss induction proof.

**Theorem:** For all natural numbers $n$, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

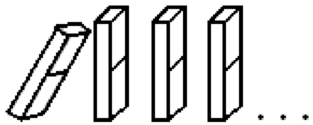
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$
\begin{aligned}
1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

$P(k+1)$! By principle of induction... □

## Notes visualization

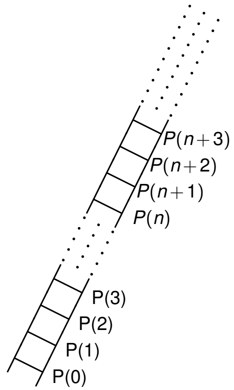
Note's visualization: an infinite sequence of dominos.



Prove they all fall down;

- $P(0)$ = "First domino falls"
- $(\forall k)\ P(k) \implies P(k+1)$:
  "$k$th domino falls implies that $k+1$st domino falls"

## Climb an infinite ladder?




$$
\begin{gathered}
P(0) \\
\forall k, P(k) \implies P(k+1) \\
P(0) \implies P(1) \implies P(2) \implies P(3) \ldots \\
(\forall n \in N) P(n)
\end{gathered}
$$

Your favorite example of forever..or the natural numbers...

## Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

**Induction Step.** $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$
...
true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$
...

Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

## Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.

The basic form

- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, "Induction Hypothesis"
  - Prove $P(k+1)$. "Induction Step."

$P(n)$ true for all natural numbers $n$!!!
Get to use $P(k)$ to prove $P(k+1)$! ! ! !

## Next Time.

More induction!
See you on Tuesday!