

CS70: Lecture 2. Outline.

Today: Proofs!!!

1. By Example.
2. Direct. (Prove $P \implies Q$.)
3. by Contraposition (Prove $P \implies Q$)
4. by Contradiction (Prove P .)
5. by Cases

If time: discuss induction.

Quick Background and Notation.

Integers closed under addition.

$$a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z}$$

$a|b$ means “a divides b”.

2|4? Yes! Since for $q = 2$, $4 = (2)2$.

7|23? No! No q where true.

4|2? No!

Formally: $a|b \iff \exists q \in \mathbb{Z}$ where $b = aq$.

3|15 since for $q = 5$, $15 = 3(5)$.

A natural number $p > 1$, is **prime** if it is divisible only by 1 and itself.

Direct Proof.

Theorem: For any $a, b, c \in Z$, if $a|b$ and $a|c$ then $a|(b - c)$.

Proof: Assume $a|b$ and $a|c$

$$b = aq \text{ and } c = aq' \text{ where } q, q' \in Z$$

$$b - c = aq - aq' = a(q - q') \text{ Done?}$$

$(b - c) = a(q - q')$ and $(q - q')$ is an integer so

$$a|(b - c)$$



Works for $\forall a, b, c$?

Argument applies to *every* $a, b, c \in Z$.

Direct Proof Form:

Goal: $P \implies Q$

Assume P .

...

Therefore Q .

Another direct proof.

Let D_3 be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of n is divisible by 11, then $11|n$.

$$\forall n \in D_3, (11 | \text{alt. sum of digits of } n) \implies 11 | n$$

Examples:

$n = 121$ Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$ Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some a, b, c .

Assume: Alt. sum: $a - b + c = 11k$ for some integer k .

Add $99a + 11b$ to both sides.

$$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$$

Left hand side is n , $k + 9a + b$ is integer. $\implies 11|n$. □

Direct proof of $P \implies Q$:

Assumed P : $11|a - b + c$. Proved Q : $11|n$.

The Converse

Thm: $\forall n \in D_3, (11 | \text{alt. sum of digits of } n) \implies 11 | n$

Is converse a theorem?

$\forall n \in D_3, (11 | n) \implies (11 | \text{alt. sum of digits of } n)$

Yes? No?

Another Direct Proof.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Proof: Assume $11|n$.

$$n = 100a + 10b + c = 11k \implies$$

$$99a + 11b + (a - b + c) = 11k \implies$$

$$a - b + c = 11k - 99a - 11b \implies$$

$$a - b + c = 11(k - 9a - b) \implies$$

$$a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}$$

That is $11|\text{alternating sum of digits}$. □

Note: similar proof to other. In this case every \implies is \iff

Often works with arithmetic properties ...

...**not** when multiplying by 0.

We have.

Theorem: $\forall n \in N', (11|\text{alt. sum of digits of } n) \iff (11|n)$

Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If n is odd then d is odd.

$n = 2k + 1$ what do we know about d ?

What to do? Is it **even** true?

Hey, that rhymes ...and there is a pun ... colored blue.
Anyway, what to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$

...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$.

Proof: Assume $\neg Q$: d is even. $d = 2k$.

$d|n$ so we have

$$n = qd = q(2k) = 2(kq)$$

n is even. $\neg P$



Another Contraposition...

Lemma: For every n in N , n^2 is even $\implies n$ is even. ($P \implies Q$)

n^2 is even, $n^2 = 2k$, ... $\sqrt{2k}$ even?

Proof by contraposition: ($P \implies Q$) \equiv ($\neg Q \implies \neg P$)

$P =$ ' n^2 is even.' $\neg P =$ ' n^2 is odd'

$Q =$ ' n is even' $\neg Q =$ ' n is odd'

Prove $\neg Q \implies \neg P$: n is odd $\implies n^2$ is odd.

$$n = 2k + 1$$

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

$n^2 = 2l + 1$ where l is a natural number..

... and n^2 is odd!

$\neg Q \implies \neg P$ so $P \implies Q$ and ...



Proof by contradiction: form

Theorem: $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

Theorem: P .

$$\neg P \implies P_1 \cdots \implies R$$

$$\neg P \implies Q_1 \cdots \implies \neg R$$

$$\neg P \implies R \wedge \neg R \equiv \text{False}$$

$$\text{or } \neg P \implies \text{False}$$

Contrapositive of $\neg P \implies \text{False}$ is $\text{True} \implies P$.

Theorem P is proven. □

Contradiction

Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: **a and b have no common factors.**

$$\sqrt{2}b = a$$

$$2b^2 = a^2 = 4k^2$$

a^2 is even $\implies a$ is even.

$a = 2k$ for some integer k

$$b^2 = 2k^2$$

b^2 is even $\implies b$ is even.

a and b have a common factor. Contradiction.



Proof by contradiction: example

Theorem: There are infinitely many primes.

Proof:

- ▶ Assume finitely many primes: p_1, \dots, p_k .
- ▶ Consider number

$$q = (p_1 \times p_2 \times \dots \times p_k) + 1.$$

- ▶ q cannot be one of the primes as it is larger than any p_i .
- ▶ q has prime divisor p (" $p > 1$ " = R) which is one of p_i .
- ▶ p divides both $x = p_1 \cdot p_2 \cdot \dots \cdot p_k$ and q , and divides $x - q$,
- ▶ $\implies p \mid x - q \implies p \leq x - q = 1$.
- ▶ so $p \leq 1$. (**Contradicts R .**)

The original assumption that "the theorem is false" is false, thus the theorem is proven.



Product of first k primes..

Did we prove?

- ▶ “The product of the first k primes plus 1 is prime.”
- ▶ No.
- ▶ The chain of reasoning started with a false statement.

Consider example..

- ▶ $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
- ▶ There is a prime *in between* 13 and $q = 30031$ that divides q .
- ▶ Proof assumed no primes *in between* p_k and q .

Proof by cases.

Theorem: $x^5 - x + 1 = 0$ has no solution in the rationals.

Proof: First a lemma...

Lemma: If x is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both a and b are even.

Reduced form $\frac{a}{b}$: a and b can't both be even! + Lemma
 \implies no rational solution. □

Proof of lemma: Assume a solution of the form a/b .

$$\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0$$

Multiply by b^5 ,

$$a^5 - ab^4 + b^5 = 0$$

Case 1: a odd, b odd: odd - odd + odd = even. **Not possible.**

Case 2: a even, b odd: even - even + odd = odd. **Not possible.**

Case 3: a odd, b even: odd - even + even = odd. **Not possible.**

Case 4: a even, b even: even - even + even = even. **Possible.**

The fourth case is the only one possible, so the lemma follows. □

Proof by cases.

Theorem: There exist irrational x and y such that x^y is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

▶ New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

▶

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2.$$

Thus, we have irrational x and y with a rational x^y (i.e., 2).

One of the cases is true so theorem holds. □

Question: Which case holds? Don't know!!!

Be careful.

Theorem: $3 = 4$

Proof: Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get
 $4 = 3$.

By commutativity theorem holds. □

Don't assume what you want to prove!

Be really careful!

Theorem: $1 = 2$

Proof: For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$

$$x(x - y) = (x + y)(x - y)$$

$$x = (x + y)$$

$$x = 2x$$

$$1 = 2$$



Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

$P \implies Q$ does not mean $Q \implies P$.

Summary: Note 2.

Direct Proof:

To Prove: $P \implies Q$. Assume P . Prove Q .

By Contraposition:

To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:

To Prove: P Assume $\neg P$. Prove **False** .

By Cases: informal.

Universal: show that statement holds in all cases.

Existence: used cases where one is true.

Either $\sqrt{2}$ and $\sqrt{2}$ worked.

or $\sqrt{2}$ and $\sqrt{2}\sqrt{2}$ worked.

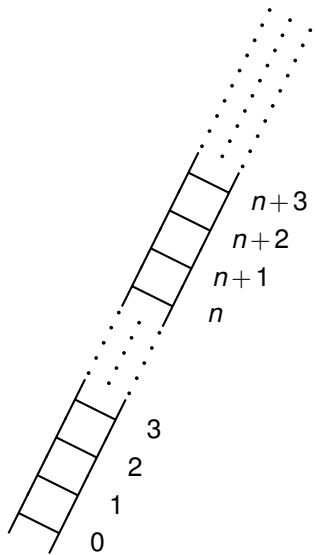
Careful when proving!

Don't assume the theorem. Divide by zero. Watch converse. ...

CS70: Note 3. Induction!

1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.

The natural numbers.



0, 1, 2, 3,
..., n , $n+1$, $n+2$, $n+3$, ...

A formula.

Teacher: Hello class.

Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in \mathbf{N}) : \sum_{i=0}^n i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

$$\forall(n \in \mathbf{N}) : P(n).$$

$P(n)$ is " $\sum_{i=0}^n i = \frac{(n)(n+1)}{2}$ ".

Principle of Induction:

- ▶ Prove $P(0)$.
- ▶ Assume $P(k)$, "Induction Hypothesis"
- ▶ Prove $P(k+1)$. "Induction Step."

Gauss induction proof.

Theorem: For all natural numbers n , $0 + 1 + 2 \cdots n = \frac{n(n+1)}{2}$

Base Case: Does $0 = \frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

Induction Hypothesis: $P(k) = 1 + \cdots + k = \frac{k(k+1)}{2}$

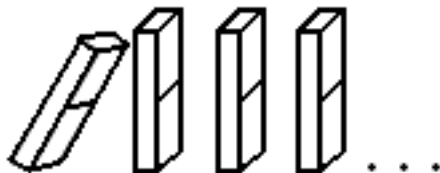
$$\begin{aligned} 1 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k^2 + k + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

$P(k+1)$! By principle of induction...



Notes visualization

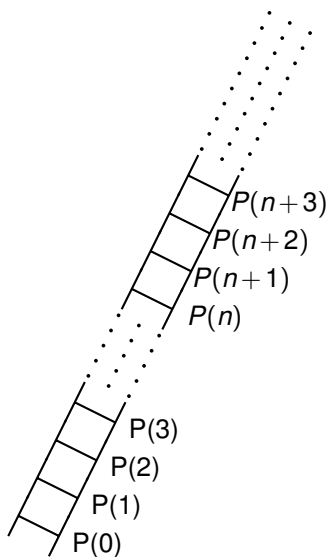
Note's visualization: an infinite sequence of dominos.



Prove they all fall down;

- ▶ $P(0)$ = “First domino falls”
- ▶ $(\forall k) P(k) \implies P(k+1)$:
“ k th domino falls implies that $k+1$ st domino falls”

Climb an infinite ladder?



$$\begin{aligned} & P(0) \\ & \forall k, P(k) \implies P(k+1) \\ P(0) \implies & P(1) \implies P(2) \implies P(3) \dots \\ & (\forall n \in \mathbb{N}) P(n) \end{aligned}$$

Your favorite example of forever..or the natural numbers...

Gauss and Induction

Child Gauss: $(\forall n \in \mathbf{N})(\sum_{i=1}^n i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^k i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k + 1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^k i) + (k+1) = \frac{k(k+1)}{2} + k+1 = \frac{(k+1)(k+2)}{2}.$$

How about $k + 2$. Same argument starting at $k + 1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^0 i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step \implies true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step \implies true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k \implies$ true for $n = k + 1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

...

Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbf{N})(P(k))$$

- ▶ For all natural numbers n , $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- ▶ For all $n \in \mathbf{N}$, $n^3 - n$ is divisible by 3.
- ▶ The sum of the first n odd integers is a perfect square.

The basic form

- ▶ Prove $P(0)$. “Base Case”.
- ▶ $P(k) \implies P(k+1)$
 - ▶ Assume $P(k)$, “Induction Hypothesis”
 - ▶ Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers $n!!!$

Get to use $P(k)$ to prove $P(k+1)!!!!$

Next Time.

More induction!
See you on Tuesday!