Review.

Theory: If you drink you must be at least 18.

Which cards do you turn over?

\[
\begin{align*}
\text{Drink} &= \geq 18 \\
\text{\textless 18} &= \Rightarrow \text{Don't Drink.}
\end{align*}
\]
Review.

Theory: If you drink you must be at least 18.
Which cards do you turn over?
Theory: If you drink you must be at least 18.

Which cards do you turn over?

Drink $\implies \geq 18$
Theory: If you drink you must be at least 18.

Which cards do you turn over?

Drink $\implies \geq 18$

"$< 18$" $\implies$ Don’t Drink.
Review.

Theory: If you drink you must be at least 18.
Which cards do you turn over?
Drink $\implies \geq 18$
"$< 18$" $\implies$ Don’t Drink. Contrapositive.
Review.

Theory: If you drink you must be at least 18.

Which cards do you turn over?

Drink $\implies \ "\geq 18"$

"$< 18" \implies$ Don’t Drink. Contrapositive.

$\land, \lor, \neg, \ P \implies Q \equiv \neg P \lor Q.$
Theory: If you drink you must be at least 18.

Which cards do you turn over?

Drink $\implies \geq 18$

"$< 18$" $\implies$ Don’t Drink. Contrapositive.

$\land, \lor, \neg, P \implies Q \equiv \neg P \lor Q$.

Truth Table. Putting together identities. (E.g., cases, substitution.)
Today: Proofs!!!

1. By Example.
2. Direct. (Prove $P \implies Q$.)
3. by Contraposition (Prove $P \implies Q$)
4. by Contradiction (Prove $P$.)
5. by Cases

If time: discuss induction.
Quick Background and Notation.

Integers closed under addition.

--

$a, b \in \mathbb{Z} \Rightarrow a + b \in \mathbb{Z}$

$a \mid b$ means "$a$ divides $b$".

$2 \mid 4$?
Yes! Since for $q = 2$, $4 = (2)2$.

$7 \mid 23$?
No! No $q$ where true.

$4 \mid 2$?
No!

Formally: $a \mid b \iff \exists q \in \mathbb{Z}$ where $b = aq$.

$3 \mid 15$ since for $q = 5$, $15 = 3(5)$.

A natural number $p > 1$, is prime if it is divisible only by 1 and itself.
Quick Background and Notation.

Integers closed under addition.

\[ a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z} \]
Quick Background and Notation.

Integers closed under addition.

\[ a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z} \]

\( a \mid b \) means “a divides b”.

2 | 4? Yes!

Since for \( q = 2, 4 = (2)^2 \).

7 | 23? No!

4 | 2? No!

 Poll

Formally:

\[ a \mid b \iff \exists q \in \mathbb{Z} \text{ where } b = aq . \]

3 | 15 since for \( q = 5, 15 = 3(5) \).

A natural number \( p > 1 \), is prime if it is divisible only by 1 and itself.
Quick Background and Notation.

Integers closed under addition.

\[ a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z} \]

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2 \( \mid \) 4?

2 \( \mid \) 15 since for \( q = 5 \), 15 = 3 \( \times \) 5.
Quick Background and Notation.

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2\( \mid \)4?

3\( \mid \)15 since for \( q = 5 \), 

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2|4?

7|23?
Quick Background and Notation.

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\[ a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z} \]

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2\|4? 
7\|23? 
4\|2?
Integers closed under addition.

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2|4?  
7|23?  
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3\mid15
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A natural number \( p > 1 \), is \textbf{prime} if it is divisible only by 1 and itself.
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b - c)$.
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid (b − c)$.

**Proof:** Assume $a \mid b$ and $a \mid c$
Theorem: For any $a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid (b - c)$.

Proof: Assume $a \mid b$ and $a \mid c$

\[ b = aq \]
Theorem: For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b - c)$.

Proof: Assume $a|b$ and $a|c$

$b = aq$ and $c = aq'$

$$(b - c) = aq - aq' = a(q - q')$$

where $q - q' \in \mathbb{Z}$. This shows that $a|(b - c)$.
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b - c)$.

**Proof:** Assume $a|b$ and $a|c$

$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$
Theorem: For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b - c)$.

Proof: Assume $a|b$ and $a|c$

   $b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$

   $b - c = aq - aq'$
Direct Proof.

**Theorem:** For any \( a, b, c \in \mathbb{Z} \), if \( a | b \) and \( a | c \) then \( a | (b - c) \).

**Proof:** Assume \( a | b \) and \( a | c \)

\[
b = aq \quad \text{and} \quad c = aq' \quad \text{where} \quad q, q' \in \mathbb{Z}
\]

\[
b - c = aq - aq' = a(q - q')
\]
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a | b$ and $a | c$ then $a | (b - c)$.

**Proof:** Assume $a | b$ and $a | c$

$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$

$b - c = aq - aq' = a(q - q')$ Done?

Used distributive property and definition of divides.
Direct Proof.

Theorem: For any $a, b, c \in \mathbb{Z}$, if $a | b$ and $a | c$ then $a | (b - c)$.

Proof: Assume $a | b$ and $a | c$

$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$

$b - c = aq - aq' = a(q - q')$ Done?

$(b - c) = a(q - q')$
Theorem: For any $a, b, c \in \mathbb{Z}$, if $a | b$ and $a | c$ then $a | (b - c)$.

Proof: Assume $a | b$ and $a | c$

$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$

$b - c = aq - aq' = a(q - q')$ Done?

$(b - c) = a(q - q')$ and $(q - q')$ is an integer so by definition of divides
**Theorem:** For any $a, b, c \in Z$, if $a \mid b$ and $a \mid c$ then $a \mid (b - c)$.

**Proof:** Assume $a \mid b$ and $a \mid c$

$b = aq$ and $c = aq'$ where $q, q' \in Z$

$b - c = aq - aq' = a(q - q')$ Done?

$(b - c) = a(q - q')$ and $(q - q')$ is an integer so by definition of divides $a \mid (b - c)$
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b - c)$.

**Proof:** Assume $a|b$ and $a|c$

$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$

$b - c = aq - aq' = a(q - q')$ Done?

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Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b - c)$.

**Proof:** Assume $a|b$ and $a|c$ \\
$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$ \\
$b - c = aq - aq' = a(q - q')$ Done? \\
$(b - c) = a(q - q')$ and $(q - q')$ is an integer so by definition of divides \\
$a|(b - c)$ \\
Works for $\forall a, b, c$?
Theorem: For any $a, b, c \in \mathbb{Z}$, if $a | b$ and $a | c$ then $a | (b - c)$.

Proof: Assume $a | b$ and $a | c$

\[ b = aq \quad \text{and} \quad c = aq' \quad \text{where} \quad q, q' \in \mathbb{Z} \]

\[ b - c = aq - aq' = a(q - q') \quad \text{Done?} \]

$(b - c) = a(q - q')$ and $(q - q')$ is an integer so by definition of divides $a | (b - c)$

Works for \( \forall a, b, c \)?

Argument applies to every $a, b, c \in \mathbb{Z}$.

Used distributive property and definition of divides.
**Theorem:** For any \( a, b, c \in \mathbb{Z} \), if \( a | b \) and \( a | c \) then \( a | (b - c) \).

**Proof:** Assume \( a | b \) and \( a | c \)

\[
b = aq \quad \text{and} \quad c = aq' \quad \text{where} \quad q, q' \in \mathbb{Z}
\]

\[
b - c = aq - aq' = a(q - q') \quad \text{Done?}
\]

\[
(b - c) = a(q - q') \quad \text{and} \quad (q - q') \text{ is an integer so by definition of divides}
\]

\[
a | (b - c)
\]

Works for \( \forall a, b, c \)?

Argument applies to every \( a, b, c \in \mathbb{Z} \).

Used distributive property and definition of divides.

Direct Proof Form:
**Theorem:** For any \(a, b, c \in \mathbb{Z}\), if \(a \mid b\) and \(a \mid c\) then \(a \mid (b - c)\).

**Proof:** Assume \(a \mid b\) and \(a \mid c\)

\[ b = aq \text{ and } c = aq' \text{ where } q, q' \in \mathbb{Z} \]

\[ b - c = aq - aq' = a(q - q') \text{ Done?} \]

\((b - c) = a(q - q')\) and \((q - q')\) is an integer so by definition of divides

\(a \mid (b - c)\)  \(\square\)

Works for \(\forall a, b, c\)?

Argument applies to every \(a, b, c \in \mathbb{Z}\).

Used distributive property and definition of divides.

**Direct Proof Form:**

**Goal:** \(P \implies Q\)
Direct Proof.

**Theorem:** For any $a, b, c \in Z$, if $a|b$ and $a|c$ then $a|(b − c)$.

**Proof:** Assume $a|b$ and $a|c$

$b = aq$ and $c = aq'$ where $q, q' \in Z$

$b − c = aq − aq' = a(q − q')$ Done?

$(b − c) = a(q − q')$ and $(q − q')$ is an integer so by definition of divides

$a|(b − c)$

Works for $\forall a, b, c$?

Argument applies to every $a, b, c \in Z$.

Used distributive property and definition of divides.

Direct Proof Form:

Goal: $P \implies Q$

Assume $P$.  

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a | b$ and $a | c$ then $a | (b - c)$.

**Proof:** Assume $a | b$ and $a | c$

\[ b = aq \] and \[ c = aq' \] where $q, q' \in \mathbb{Z}$

\[ b - c = aq - aq' = a(q - q') \] Done?

$(b - c) = a(q - q')$ and $(q - q')$ is an integer so by definition of divides

\[ a | (b - c) \]

Works for $\forall a, b, c$?

Argument applies to every $a, b, c \in \mathbb{Z}$.

Used distributive property and definition of divides.

**Direct Proof Form:**

**Goal:** $P \implies Q$

Assume $P$.

...
**Theorem:** For any \(a, b, c \in \mathbb{Z}\), if \(a \mid b\) and \(a \mid c\) then \(a \mid (b - c)\).

**Proof:** Assume \(a \mid b\) and \(a \mid c\)

\[
b = aq \quad \text{and} \quad c = aq' \quad \text{where} \quad q, q' \in \mathbb{Z}
\]

\[
b - c = aq - aq' = a(q - q') \quad \text{Done?}
\]

\[
(b - c) = a(q - q') \quad \text{and} \quad (q - q') \text{ is an integer so by definition of divides}
\]

\[
a \mid (b - c)
\]

Works for \(\forall a, b, c\)?

Argument applies to every \(a, b, c \in \mathbb{Z}\).

Used distributive property and definition of divides.

**Direct Proof Form:**

**Goal:** \(P \implies Q\)

**Assume** \(P\).

\[
\ldots
\]

**Therefore** \(Q\).
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11 | n$. 

Examples:

- $n = 121$  
  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

- $n = 605$  
  Alt Sum: $6 - 0 + 5 = 11$. Divis. by 11. As is 605 = 11(55).

Proof:

For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add 99$a + 11b$ to both sides.

$100a + 10b + c = 11k + 99a + 11b$

The left hand side is $n$, $k + 9a + b$ is integer.

$⇒ 11 | n$. 

Direct proof of $P =⇒ Q$:

Assumed $P$: $11 | a - b + c$.

Proved $Q$: $11 | n$. 
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11 \mid n$. 

Examples:

$\begin{align*}
\text{n} &= 121 \\
\text{Alt Sum} &= 1 - 2 + 1 = 0. \\
\text{Divis. by 11. As is 121.}
\end{align*}$

$\begin{align*}
\text{n} &= 605 \\
\text{Alt Sum} &= 6 - 0 + 5 = 11. \\
\text{Divis. by 11. As is 605.}
\end{align*}$

Proof:

For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$

Left hand side is $n$, $k + 9a + b$ is integer.

$\Rightarrow 11 \mid n$. 

Direct proof of $P = \Rightarrow Q$:

Assumed $P$: $11 \mid a - b + c$.

Proved $Q$: $11 \mid n$. 
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

$$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$$
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11|n$.

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Examples:
$n = 121 \quad \text{Alt Sum: } 1 - 2 + 1 = 0$. 
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

\[
\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n
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Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11.
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11 \mid n$.

\[ \forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n \]

Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11 | n$.

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Examples:
\[ n = 121 \quad \text{Alt Sum: } 1 - 2 + 1 = 0. \text{ Divis. by 11. As is 121.} \]
\[ n = 605 \]
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11|n$.

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Examples:
- $n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.
- $n = 605$  Alt Sum: $6 - 0 + 5 = 11$
Another direct proof.

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$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divisible by 11. As is 121.

$n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divisible by 11.
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Proof: For $n \in D_3$, 
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

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Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$  Alt Sum: $6 - 0 + 5 = 11$  Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$. 
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Examples:
- $n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.
- $n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum:
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

\[ \forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n \]

Examples:
\begin{align*}
  n &= 121 \quad \text{Alt Sum: } 1 - 2 + 1 = 0. \text{ Divis. by 11. As is 121.} \\
  n &= 605 \quad \text{Alt Sum: } 6 - 0 + 5 = 11 \text{ Divis. by 11. As is } 605 = 11(55)
\end{align*}

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c$
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11|n$.

$$
\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n
$$

Examples:
$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.
$n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

**Proof:** For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$. 
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11 | n$.

$$\forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 | n$$

Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11|n$.

$$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$$

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Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.
Add $99a + 11b$ to both sides.
$$100a + 10b + c = 11k + 99a + 11b$$
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

\[ \forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n \]

Examples:

$n = 121$ \hspace{1em} Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$ \hspace{1em} Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

\[ 100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b) \]
Another direct proof.

Let \( D_3 \) be the 3 digit natural numbers.

Theorem: For \( n \in D_3 \), if the alternating sum of digits of \( n \) is divisible by 11, than \( 11 \mid n \).

\[ \forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n \]

Examples:
\( n = 121 \)  Alt Sum: \( 1 - 2 + 1 = 0 \). Divis. by 11. As is 121.
\( n = 605 \)  Alt Sum: \( 6 - 0 + 5 = 11 \) Divis. by 11. As is 605 = 11(55)

Proof: For \( n \in D_3 \), \( n = 100a + 10b + c \), for some \( a, b, c \).
Assume: Alt. sum: \( a - b + c = 11k \) for some integer \( k \).
Add \( 99a + 11b \) to both sides.
\[ 100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b) \]
Left hand side is \( n \),
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

$$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$$

Examples:

- $n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.
- $n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$$

Left hand side is $n$, $k + 9a + b$ is integer.
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11 | n$.

$$\forall n \in D_3, (11 | \text{alt. sum of digits of } n) \implies 11 | n$$

Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

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Add $99a + 11b$ to both sides.

$$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$$

Left hand side is $n$, $k + 9a + b$ is integer.  $\implies 11 | n$. 
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11|n$.

$$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$$

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Add $99a + 11b$ to both sides.

$$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$$

Left hand side is $n$, $k + 9a + b$ is integer.  $\implies 11|n$.  $\blacksquare$
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

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Examples:
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Add $99a + 11b$ to both sides.

$$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$$

Left hand side is $n$, $k + 9a + b$ is integer.  $\implies 11|n$. \hfill \Box$

Direct proof of $P \implies Q$:
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$

Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$

Left hand side is $n$, $k + 9a + b$ is integer.  $\implies 11|n$.  \hfill \Box$

Direct proof of $P \implies Q$:

Assumed $P$: $11|a - b + c$.
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11|n$.

$$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$$

Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$$

Left hand side is $n$, $k + 9a + b$ is integer. $\implies 11|n$. □

Direct proof of $P \implies Q$:

Assumed $P$: $11|a - b + c$. Proved $Q$: $11|n$. 
The Converse

Thm: \( \forall n \in D_3, (11 | \text{alt. sum of digits of } n) \implies 11 | n \)
The Converse

Thm: $\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$

Is converse a theorem?
$\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$
The Converse

Thm: $\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$

Is converse a theorem?

$\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Yes?
The Converse

Thm: \( \forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n \)

Is converse a theorem?
\( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Yes? No?
The Converse

Thm: $\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$

Is converse a theorem?
$\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Yes? No?
Poll
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \iff (11|\text{alt. sum of digits of } n) \)
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11 | n) \iff (11 | \text{alt. sum of digits of } n) \)

Proof:

Assume \( 11 | n \).

\[ n = 100a + 10b + c = 11k \]

\[ 99a + 11b + (a - b + c) = 11k \]

\[ a - b + c = 11(k - 9a - b) \]

That is \( 11 | \text{alternating sum of digits.} \)

Note: similar proof to other. In this case every \( \Rightarrow \) is \( \iff \).

Often works with arithmetic properties ...

...not when multiplying by 0.
We have.

Theorem: \( \forall n \in \mathbb{N}', (11 | n) \iff (11 | \text{alt. sum of digits of } n) \)
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume 11|\( n \).
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume \( 11|n \).

\[
n = 100a + 10b + c = 11k
\]
Another Direct Proof.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Proof: Assume $11|n$.

$n = 100a + 10b + c = 11k \implies 99a + 11b + (a - b + c) = 11k$
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume \(11|n\).

\[
n = 100a + 10b + c = 11k \implies 99a + 11b + (a - b + c) = 11k \implies a - b + c = 11k - 99a - 11b
\]
Another Direct Proof.

Theorem:  \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume 11|n.

\[
\begin{align*}
n &= 100a + 10b + c = 11k \implies \\
99a + 11b + (a - b + c) &= 11k \implies \\
a - b + c &= 11k - 99a - 11b \implies \\
a - b + c &= 11(k - 9a - b)
\end{align*}
\]
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume 11\( |n \).

\[
n = 100a + 10b + c = 11k \implies
99a + 11b + (a - b + c) = 11k \implies
a - b + c = 11k - 99a - 11b \implies
a - b + c = 11(k - 9a - b) \implies
a - b + c = 11\ell
\]
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \iff (11|\text{alt. sum of digits of } n) \)

**Proof:** Assume \( 11|n \).

\[
\begin{align*}
n &= 100a + 10b + c = 11k \\ 
99a + 11b + (a - b + c) &= 11k \\
     a - b + c &= 11k - 99a - 11b \\
             a - b + c &= 11(k - 9a - b) \\
             a - b + c &= 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}
\end{align*}
\]
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume \( 11|n \).

\begin{align*}
n &= 100a + 10b + c = 11k \implies \\
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a - b + c &= 11k - 99a - 11b \implies \\
a - b + c &= 11(k - 9a - b) \implies \\
a - b + c &= 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}
\end{align*}

That is \( 11|\text{alternating sum of digits} \). \( \square \)
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11\mid n) \implies (11\mid \text{alt. sum of digits of } n) \)

Proof: Assume \( 11\mid n \).

\[ n = 100a + 10b + c = 11k \implies \]
\[ 99a + 11b + (a - b + c) = 11k \implies \]
\[ a - b + c = 11k - 99a - 11b \implies \]
\[ a - b + c = 11(k - 9a - b) \implies \]
\[ a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z} \]

That is \( 11\mid \text{alternating sum of digits.} \)

Note: similar proof to other. In this case every \( \implies \) is \( \iff \)
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume \( 11|n \).

\[
n = 100a + 10b + c = 11k \implies \\
99a + 11b + (a - b + c) = 11k \implies \\
a - b + c = 11(k - 9a - b) \implies \\
\]
That is \( 11|\text{alternating sum of digits} \).

Note: similar proof to other. In this case every \( \implies \) is \( \iff \)

Often works with arithmetic properties...
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11 \mid n) \implies (11 \mid \text{alt. sum of digits of } n) \)

**Proof:** Assume \( 11 \mid n \).

\[
 n = 100a + 10b + c = 11k 
\]
\[
 99a + 11b + (a - b + c) = 11k 
\]
\[
 a - b + c = 11k - 99a - 11b 
\]
\[
 a - b + c = 11(k - 9a - b) 
\]
\[
 a - b + c = 11 \ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z} 
\]

That is \( 11 \mid \text{alternating sum of digits} \).

Note: similar proof to other. In this case every \( \implies \) is \( \iff \)

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...not when multiplying by 0.
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume \(11|n\).

\[
n = 100a + 10b + c = 11k \implies 99a + 11b + (a - b + c) = 11k \implies a - b + c = 11k - 99a - 11b \implies a - b + c = 11(k - 9a - b) \implies a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}
\]

That is \(11|\text{alternating sum of digits.} \)

Note: similar proof to other. In this case every \(\implies\) is \(\iff\)

Often works with arithmetic properties ...

...not when multiplying by 0.

We have.
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

**Proof:** Assume \( 11|n \).

\[
\begin{align*}
n &= 100a + 10b + c = 11k \\ 99a + 11b + (a - b + c) &= 11k \\ a - b + c &= 11k - 99a - 11b \\ a - b + c &= 11(k - 9a - b) \\ a - b + c &= 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}
\end{align*}
\]

That is \( 11|\text{alternating sum of digits} \). \( \Box \)

Note: similar proof to other. In this case every \( \implies \) is \( \iff \)

Often works with arithmetic properties ...

...not when multiplying by 0.

We have.

Theorem: \( \forall n \in N', (11|\text{alt. sum of digits of } n) \iff (11|n) \)
Proof by Contraposition

**Theorem:** For $n \in \mathbb{Z}^+$ and $d \mid n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$.

What do we know about $d$?

What to do?

Is it even true?

Hey, that rhymes... and there is a pun... colored blue.

Anyway, what to do?

**Goal:** Prove $P \implies Q$.

Assume $\neg Q$... and prove $\neg P$.

**Conclusion:** $\neg Q \implies \neg P$.

**Proof:** Assume $\neg Q$: $d$ is even.

$d = 2k$.

$d \mid n$ so we have $n = qd = q(2k) = 2(kq)$.

$n$ is even. $\neg P$.
Proof by Contraposition

Thm: For $n \in Z^+$ and $d \mid n$. If $n$ is odd then $d$ is odd.
Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[
n = 2k + 1
\]
Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \text{ what do we know about } d? \]
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do?
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d | n \). If \( n \) is odd then \( d \) is odd.

\[
n = 2k + 1 \quad \text{what do we know about } d? 
\]

What to do? Is it even true?
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \]

what do we know about \( d \)?

What to do? Is it even true?

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Proof by Contraposition

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$n = 2k + 1$ what do we know about $d$?

What to do? Is it even true?

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Proof by Contraposition

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What to do? Is it even true?

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What to do? Is it even true?

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Anyway, what to do?

Goal: Prove $P \implies Q$. 
Proof by Contraposition

Thm: For \( n \in Z^+ \) and \( d | n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \] what do we know about \( d \)?

What to do? Is it even true?

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Goal: Prove \( P \implies Q \).
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do? Is it even true?

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Anyway, what to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do? Is it even true?

Hey, that rhymes ...and there is a pun ... colored blue.

Anyway, what to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$

...and prove $\neg P$. 
Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \] what do we know about \( d \)?

What to do? Is it even true?

Hey, that rhymes ...and there is a pun ... colored blue.

Anyway, what to do?

**Goal:** Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

**Conclusion:** \( \neg Q \implies \neg P \)
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \] what do we know about \( d \)?

What to do? Is it even true?

Hey, that rhymes ... and there is a pun ... colored blue.

Anyway, what to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

... and prove \( \neg P \).

Conclusion: \( \neg Q \implies \neg P \) equivalent to \( P \implies Q \).
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\( n = 2k + 1 \) what do we know about \( d \)?

What to do? Is it even true?

Hey, that rhymes ... and there is a pun ... colored blue.

Anyway, what to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

Conclusion: \( \neg Q \implies \neg P \) equivalent to \( P \implies Q \).

Proof: Assume \( \neg Q \): \( d \) is even.
Proof by Contraposition

Thm: For $n \in Z^+$ and $d \mid n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do? Is it even true?

Hey, that rhymes ...and there is a pun ... colored blue.

Anyway, what to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$

...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$.

Proof: Assume $\neg Q$: $d$ is even. $d = 2k$. 
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\( n = 2k + 1 \) what do we know about \( d \)?

What to do? Is it even true?

Hey, that rhymes ...and there is a pun ... colored blue.

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Proof: Assume \( \neg Q \): \( d \) is even. \( d = 2k \).

\( d \mid n \) so we have
Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If $n$ is odd then $d$ is odd.

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Proof: Assume $\neg Q$: $d$ is even. $d = 2k$.

$d|n$ so we have

$n = qd$
Thm: For $n \in Z^+$ and $d|n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do? Is it even true?

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$d|n$ so we have

$n = qd = q(2k)$
Proof by Contraposition

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\[ n = qd = q(2k) = 2(kq) \]
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$n$ is even. $\neg P$
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Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$.

Proof: Assume $\neg Q$: $d$ is even. $d = 2k$.

$d | n$ so we have

$n = qd = q(2k) = 2(kq)$

$n$ is even. $\neg P$
Lemma:
For every \( n \in \mathbb{N} \), \( n^2 \) is even \( \Rightarrow \) \( n \) is even. (\( P \Rightarrow Q \))

Proof by contraposition:
(\( P \Rightarrow Q \)) \( \equiv \) (\( \neg Q = \Rightarrow \neg P \))

\( P = \) '\( n^2 \) is even.'
\( \neg P = \) '\( n^2 \) is odd'

\( Q = \) '\( n \) is even'
\( \neg Q = \) '\( n \) is odd'

Prove \( \neg Q = \Rightarrow \neg P \):
\( n \) is odd \( \Rightarrow \) \( n^2 \) is odd.

\( n = 2k + 1 \)
\( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \)

\( n^2 = 2l + 1 \) where \( l \) is a natural number.

... and \( n^2 \) is odd!

\( \neg Q = \Rightarrow \neg P \) so \( P \Rightarrow Q \) and ...
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\iff$ $n$ is even. ($P \iff Q$)

$n^2$ is even, $n^2 = 2k$, ...

$\sqrt{2}$ even?

Proof by contraposition: ($P \iff Q$) $\equiv$ ($\neg Q = \iff \neg P$)

$P = \neg n^2$ is even.' ...........

$\neg P = \neg n^2$ is odd'

$Q = \neg n$ is even' ...........

$\neg Q = \neg n$ is odd'

Prove $\neg Q = \iff \neg P$:

$n$ is odd $\Rightarrow n^2$ is odd.

$n = 2k + 1$

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

$n^2 = 2l + 1$ where $l$ is a natural number.

... and $n^2$ is odd!

$\neg Q = \iff \neg P$ so $P = \iff Q$ and ...


Lemma: For every $n$ in $N$, $n^2$ is even $\iff n$ is even. ($P \iff Q$)

$n^2$ is even, $n^2 = 2k$, ...

... and $n^2$ is odd!

$\neg Q \implies \neg P$ so $P \implies Q$ and ...
Another Contraposition...

**Lemma:** For every $n$ in $\mathbb{N}$, $n^2$ is even $\implies$ $n$ is even. ($P \implies Q$)

$n^2$ is even, $n^2 = 2k$, ... $\sqrt{2k}$ even?
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies n$ is even. $(P \implies Q)$

**Proof by contraposition:** $(P \implies Q) \equiv (\neg Q \implies \neg P)$
Lemma: For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

Proof by contraposition: ($P \implies Q$) $\equiv$ ($\neg Q \implies \neg P$)

$P = 'n^2$ is even.' ............

\[ P = 2k \]
\[ n^2 = 4k^2 \]
\[ n^2 = 2(2k^2) + 1. \]

... and $n^2$ is odd!

$\neg Q = \implies \neg P$ so $P = \implies Q$ and ...
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

**Proof by contraposition:** ($P \implies Q) \equiv (\neg Q \implies \neg P$)

$P = \text{'}n^2\text{ is even.'} \quad \text{.........} \quad \neg P = \text{'}n^2\text{ is odd'}$
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**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

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$P = 'n^2$ is even.' ............ $\neg P = 'n^2$ is odd'$

$Q = 'n$ is even' ............
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies$ $n$ is even. ($P \implies Q$)

**Proof by contraposition:** ($P \implies Q) \equiv (\neg Q \implies \neg P$)

$P = 'n^2$ is even.$' \quad \neg P = 'n^2$ is odd'$

$Q = 'n$ is even.$' \quad \neg Q = 'n$ is odd'$

$n^2$ is even, $n^2 = 2k$, ...

$\sqrt{2k}$ even?

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd.

$n = 2k + 1$

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

$n^2 = 2l + 1$ where $l$ is a natural number.

... and $n^2$ is odd!

$\neg Q = \neg Q$ so $P = \implies Q$ and ...
Lemma: For every $n$ in $N$, $n^2$ is even $\iff$ $n$ is even. ($P \iff Q$)

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$P = \text{'}n^2$ is even.' .......... $\neg P = \text{'}n^2$ is odd'

$Q = \text{'}n$ is even' .......... $\neg Q = \text{'}n$ is odd'

Prove $\neg Q \iff \neg P$: $n$ is odd $\iff n^2$ is odd.
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Proof by contraposition: ($P \iff Q) \equiv (\neg Q \iff \neg P$)

$P =$ ’$n^2$ is even.’ ........... $\neg P =$ ’$n^2$ is odd’

$Q =$ ’$n$ is even’ ........... $\neg Q =$ ’$n$ is odd’

Prove $\neg Q \iff \neg P$: $n$ is odd $\iff n^2$ is odd.

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$P = 'n^2$ is even.' ............. $\neg P = 'n^2$ is odd'

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Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. 
Another Contraposition...

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Another Contraposition...

**Lemma:** For every \( n \) in \( N \), \( n^2 \) is even \( \implies n \) is even. \((P \implies Q)\)

**Proof by contraposition:** \((P \implies Q) \equiv (\neg Q \implies \neg P)\)

\( P = 'n^2 \text{ is even}' \) ........... \( \neg P = 'n^2 \text{ is odd}' \)

\( Q = 'n \text{ is even}' \) ........... \( \neg Q = 'n \text{ is odd}' \)

Prove \( \neg Q \implies \neg P: n \text{ is odd } \implies n^2 \text{ is odd.} \)

\( n = 2k + 1 \)

\( n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1. \)

\( n^2 = 2l + 1 \) where \( l \) is a natural number..

... and \( n^2 \) is odd!
Lemma: For every \( n \) in \( N \), \( n^2 \) is even \( \implies \) \( n \) is even. \( (P \implies Q) \)

Proof by contraposition: \( (P \implies Q) \equiv (\neg Q \implies \neg P) \)

\( P = \text{'}n^2 \text{ is even.}' \) ............. \( \neg P = \text{'}n^2 \text{ is odd}' \)

\( Q = \text{'}n \text{ is even}' \) ............. \( \neg Q = \text{'}n \text{ is odd}' \)

Prove \( \neg Q \implies \neg P \): \( n \) is odd \( \implies \) \( n^2 \) is odd.

\( n = 2k + 1 \)

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\( \neg Q \implies \neg P \)
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies n$ is even. $(P \implies Q)$

**Proof by contraposition:** $(P \implies Q) \equiv (\neg Q \implies \neg P)$

$P = 'n^2$ is even.' .......... $\neg P = 'n^2$ is odd'  
$Q = 'n$ is even' ........... $\neg Q = 'n$ is odd' 

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$\neg Q \implies \neg P$ so $P \implies Q$ and ...
Lemma: For every $n$ in $N$, $n^2$ is even $\iff$ $n$ is even. ($P \implies Q$)

Proof by contraposition: ($P \implies Q$) $\equiv$ ($\neg Q \implies \neg P$)

$P = 'n^2$ is even.' ............ $\neg P = 'n^2$ is odd'

$Q = 'n$ is even' ............ $\neg Q = 'n$ is odd'

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd.

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$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

$n^2 = 2l + 1$ where $l$ is a natural number..

... and $n^2$ is odd!

$\neg Q \implies \neg P$ so $P \implies Q$ and ...
Proof by contradiction:

**Theorem:** $\sqrt{2}$ is irrational.

**Proof:**

Must show:

For every $a, b \in \mathbb{Z}$,

$(a/b)^2 \neq 2$.

A simple property (equality) should always "not" hold.

**Proof by contradiction:**

Theorem: $\neg P \Rightarrow P \Rightarrow R \neg P \Rightarrow Q \Rightarrow \neg R \neg P \Rightarrow R \land \neg R \equiv \text{False}$

Theorem $P$ is true.

And proven.
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show:
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \),
Proof by contradiction:form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).
Proof by contradiction: form

**Theorem**: $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.
Theorem: $\sqrt{2}$ is irrational.

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Proof by contradiction:

**Theorem**: $P$. 

Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, \((\frac{a}{b})^2 \neq 2\).

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

\[ \neg P \implies P_1 \]
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in Z$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \ldots$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$
Proof by contradiction: form

**Theorem**: \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem**: \( P \).

\[ \neg P \implies P_1 \cdots \implies R \]

\[ \neg P \]
**Theorem:** $\sqrt{2}$ is irrational.

**Must show:** For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2.$

A simple property (equality) should always “not” hold.

**Proof by contradiction:**

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

1. $\neg P \implies P_1 \cdots \implies R$
2. $\neg P \implies Q_1 \cdots$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always "not" hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( (\frac{a}{b})^2 \neq 2 \).

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** \( P \).

\[-P \implies P_1 \cdots \implies R\]
\[-P \implies Q_1 \cdots \implies \neg R\]
\[-P \implies R \land \neg R\]
Proof by contradiction:form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$

$\neg P \implies R \land \neg R \equiv \text{False}$

or $\neg P \implies \text{False}$
**Proof by contradiction:**

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$

$\neg P \implies R \land \neg R \equiv \text{False}$

or $\neg P \implies \text{False}$

Contrapositive of $\neg P \implies \text{False}$ is $\text{True} \implies P$. 
Proof by contradiction:

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies Q_1 \cdots \implies \neg R$

$\neg P \implies R \land \neg R \equiv \text{False}$

or $\neg P \implies \text{False}$

Contrapositive of $\neg P \implies \text{False}$ is True $\implies P$.

Theorem $P$ is true.
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $\left(\frac{a}{b}\right)^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

\[ \neg P \implies P_1 \cdots \implies R \]

\[ \neg P \implies Q_1 \cdots \implies \neg R \]

\[ \neg P \implies R \land \neg R \equiv \text{False} \]

or $\neg P \implies \text{False}$

Contrapositive of $\neg P \implies \text{False}$ is $\text{True} \implies P$.

Theorem $P$ is true. And proven. $\square$
Theorem: $\sqrt{2}$ is irrational.
Contradiction

**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$:

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a^2b^2 = 4k^2$$

$a^2$ is even $\Rightarrow a$ is even.

$$a = 2k$$

$b^2$ is even $\Rightarrow b$ is even.

$a$ and $b$ have a common factor.

Contradiction.
Contradiction

**Theorem:** \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in Z \).

Reduced form: \( a \) and \( b \) have no common factors.

\[ \sqrt{2}b = a^2 = 4k^2 \]

\( a^2 \) is even \( \Rightarrow a \) is even.

\[ a = 2k \]

\[ b^2 = 2k^2 \]

\( b^2 \) is even \( \Rightarrow b \) is even.

\( a \) and \( b \) have a common factor.

Contradiction.
Theorem: \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = \frac{a}{b} \) for \( a, b \in \mathbb{Z} \).

Reduced form: \( a \) and \( b \) have no common factors.

Contradiction
Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in Z$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2} b = a$$
**Theorem:** \(\sqrt{2}\) is irrational.

Assume \(\neg P\): \(\sqrt{2} = a/b\) for \(a, b \in \mathbb{Z}\).

Reduction form: \(a\) and \(b\) have no common factors.

\[\sqrt{2}b = a\]

\[2b^2 = a^2\]
Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

\[
\sqrt{2} b = a
\]

\[
2b^2 = a^2
\]

$a^2$ is even $\implies$ $a$ is even.
Contradiction

**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$

$$2b^2 = a^2$$

$a^2$ is even $\iff$ $a$ is even.

$a = 2k$ for some integer $k$
Contradiction

Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$

$$2b^2 = a^2 = 4k^2$$

$a^2$ is even $\implies a$ is even.

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**Theorem:** \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in \mathbb{Z} \).

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\[
\sqrt{2}b = a
\]

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2b^2 = a^2 = 4k^2
\]

\( a^2 \) is even \( \implies \) \( a \) is even.

\( a = 2k \) for some integer \( k \)

\[
b^2 = 2k^2
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Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

\[
\sqrt{2}b = a
\]

\[
2b^2 = a^2 = 4k^2
\]

$a^2$ is even $\implies$ $a$ is even.

$a = 2k$ for some integer $k$

\[
b^2 = 2k^2
\]

$b^2$ is even $\implies$ $b$ is even.
Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$

$$2b^2 = a^2 = 4k^2$$

$a^2$ is even $\implies$ $a$ is even.

$a = 2k$ for some integer $k$

$$b^2 = 2k^2$$

$b^2$ is even $\implies$ $b$ is even.

$a$ and $b$ have a common factor.
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$a$ and $b$ have a common factor. Contradiction.
**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = \frac{a}{b}$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

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$a$ and $b$ have a common factor. Contradiction.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

Assume finitely many primes: \( p_1, \ldots, p_k \).

Consider number \( q = (p_1 \times p_2 \times \cdots \times p_k) + 1 \).

\( q \) cannot be one of the primes as it is larger than any \( p_i \).

\( q \) has prime divisor \( p \) ("\( p > 1 \) = R") which is one of \( p_i \).

\( p \) divides both \( x = p_1 \times p_2 \times \cdots \times p_k \) and \( q \), and divides \( q - x \), \( \Rightarrow p \mid q - x = \Rightarrow p \leq q - x = 1 \).

\( \Rightarrow p \leq 1 \). (Contradicts \( R \)).

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$. 

  Consider number $q = (p_1 \times p_2 \times \cdots \times p_k) + 1$.

  $q$ cannot be one of the primes as it is larger than any $p_i$.

  $q$ has prime divisor $p$ ("$p > 1" = R")$ which is one of $p_i$.

  $p$ divides both $x = p_1 \times p_2 \times \cdots \times p_k$ and $q$,

  $\Rightarrow p \mid q - x \Rightarrow p \leq q - x = 1$.

  so $p \leq 1$. ($Contradicts R$)

  The original assumption that "the theorem is false" is false,
  thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider number $q = (p_1 \times p_2 \times \cdots \times p_k) + 1$.
- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $q - x$.
- Thus, $p | q - x \Rightarrow p \leq q - x = 1$.
- So $p \leq 1$.

(Contradicts R.)

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

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$p$ divides both $x = p_1 \cdot p_2 \cdot \cdots p_k$ and $q$, and divides $q - x$.

$p | q - x \Rightarrow p | q - 1 \Rightarrow p \leq q - 1$.

so $p \leq 1$.

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**Theorem:** There are infinitely many primes.

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- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider number
  \[ q = (p_1 \times p_2 \times \cdots p_k) + 1. \]

- $q$ cannot be one of the primes as it is larger than any $p_i$.

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Proof by contradiction: example

**Theorem:** There are infinitely many primes.

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- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $q - x$.
- $\implies p | q - x$
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider number
  
  \[ q = (p_1 \times p_2 \times \cdots p_k) + 1. \]

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  - $q$ has prime divisor $p$ ("$p > 1$" = R) which is one of $p_i$.
  - $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $q - x$,
  - $\implies p | q - x \implies p \leq q - x$
Proof by contradiction: example

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- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $q - x$,
  
  $$\implies p | q - x \implies p \leq q - x = 1.$$  

- so $p \leq 1$.  

The original assumption that "the theorem is false" is false, thus the theorem is proven.
**Proof by contradiction: example**

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- $p$ divides both $x = p_1 \cdot p_2 \cdot \cdots \cdot p_k$ and $q$, and divides $q - x$,
- $\implies p|q - x \implies p \leq q - x = 1$.
- so $p \leq 1$. (**Contradicts** $R.$)
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider number $q = (p_1 \times p_2 \times \cdots p_k) + 1$.

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$p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $q - x$,
\[ \implies p|q - x \implies p \leq q - x = 1. \]
so $p \leq 1$. (Contradicts R.)

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

▶ Assume finitely many primes: \(p_1, \ldots, p_k\).

▶ Consider number \(q = (p_1 \times p_2 \times \cdots p_k) + 1\).

▶ \(q\) cannot be one of the primes as it is larger than any \(p_i\).

▶ \(q\) has prime divisor \(p\) (\(p > 1\) = \(R\)) which is one of \(p_i\).

▶ \(p\) divides both \(x = p_1 \cdot p_2 \cdots p_k\) and \(q\), and divides \(q - x\),

▶ \(\Rightarrow p|q - x \Rightarrow p \leq q - x = 1\).

▶ so \(p \leq 1\). (Contradicts \(R\).)

The original assumption that “the theorem is false” is false, thus the theorem is proven.
Product of first $k$ primes.

Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”

- There is a prime in between 13 and $q = 30031$ that divides $q$.

- Proof assumed no primes in between $p_k$ and $q$. 
Product of first $k$ primes..

Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
- No.
Product of first $k$ primes.

Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.
Product of first $k$ primes..

Did we prove?

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- No.
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Consider example..
Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.

Consider example..

- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
Did we prove?
▶ “The product of the first $k$ primes plus 1 is prime.”
▶ No.
▶ The chain of reasoning started with a false statement.

Consider example..
▶ $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
▶ There is a prime *in between* 13 and $q = 30031$ that divides $q$. 
Did we prove?
- “The product of the first $k$ primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.

Consider example..
- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
- There is a prime \textit{in between} 13 and $q = 30031$ that divides $q$.
- Proof assumed no primes \textit{in between} $p_k$ and $q$. 
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

---

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $a/b$: $a$ and $b$ can't both be even! Therefore, $\Rightarrow$ no rational solution.

**Proof of lemma:**

Assume a solution of the form $a/b$.

$$
(a/b)^5 - a/b + 1 = 0
$$

Multiply by $b^5$,

$$
a^5 - ab^4 + b^5 = 0
$$

Case 1: $a$ odd, $b$ odd: odd - odd + odd = even. Not possible.

Case 2: $a$ even, $b$ odd: even - even + odd = even. Not possible.

Case 3: $a$ odd, $b$ even: odd - even + even = even. Not possible.

Case 4: $a$ even, $b$ even: even - even + even = even. Possible.

The fourth case is the only one possible, so the lemma follows.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.
**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even!
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma $\implies$ no rational solution.
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = \frac{a}{b} \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( a/b \).
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( a/b \).

\[
\left( \frac{a}{b} \right)^5 - \frac{a}{b} + 1 = 0
\]
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = \frac{a}{b} \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( \frac{a}{b} \).

\[
\left( \frac{a}{b} \right)^5 - \frac{a}{b} + 1 = 0
\]

Multiply by \( b^5 \),

\[
a^5 - ab^4 + b^5 = 0
\]
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( a/b \).

\[
\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0
\]

Multiply by \( b^5 \),

\[
a^5 - ab^4 + b^5 = 0
\]

Case 1: \( a \) odd, \( b \) odd: odd - odd + odd = even.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can't both be even! + Lemma $\implies$ no rational solution.

**Proof of lemma:** Assume a solution of the form $a/b$.

$$\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0$$

Multiply by $b^5$,

$$a^5 - ab^4 + b^5 = 0$$

Case 1: $a$ odd, $b$ odd: odd - odd +odd = even. Not possible.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma $\implies$ no rational solution.

**Proof of lemma:** Assume a solution of the form $a/b$.

$$ \left( \frac{a}{b} \right)^5 - \frac{a}{b} + 1 = 0 $$

Multiply by $b^5$,

$$ a^5 - ab^4 + b^5 = 0 $$

Case 1: $a$ odd, $b$ odd: odd - odd +odd = even. Not possible.

Case 2: $a$ even, $b$ odd: even - even +odd = even.
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), *then both \( a \) and \( b \) are even.*

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( a/b \).

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\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0
\]

Multiply by \( b^5 \),

\[
a^5 - ab^4 + b^5 = 0
\]

Case 1: \( a \) odd, \( b \) odd: odd - odd +odd = even. *Not possible.*

Case 2: \( a \) even, \( b \) odd: even - even +odd = even. *Not possible.*
Proof by cases.

**Theorem:** \(x^5 - x + 1 = 0\) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \(x\) is a solution to \(x^5 - x + 1 = 0\) and \(x = a/b\) for \(a, b \in \mathbb{Z}\), then both \(a\) and \(b\) are even.

Reduced form \(\frac{a}{b}\): \(a\) and \(b\) can't both be even! + Lemma \(\implies\) no rational solution.

**Proof of lemma:** Assume a solution of the form \(a/b\).

\[
\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0
\]

Multiply by \(b^5\),

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a^5 - ab^4 + b^5 = 0
\]

Case 1: \(a\) odd, \(b\) odd: \(\text{odd - odd +odd = even. Not possible.}\)

Case 2: \(a\) even, \(b\) odd: \(\text{even - even +odd = even. Not possible.}\)

Case 3: \(a\) odd, \(b\) even: \(\text{odd - even +even = even.}\)
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

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Multiply by \( b^5 \),

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Case 1: \( a \) odd, \( b \) odd: odd - odd +odd = even. Not possible.
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Case 3: \( a \) odd, \( b \) even: odd - even +even = even. Not possible.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can't both be even! + Lemma $\implies$ no rational solution.

**Proof of lemma:** Assume a solution of the form $a/b$.

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\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0
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Multiply by $b^5$,

\[
a^5 - ab^4 + b^5 = 0
\]

Case 1: $a$ odd, $b$ odd: odd - odd + odd = even. Not possible.
Case 2: $a$ even, $b$ odd: even - even + odd = even. Not possible.
Case 3: $a$ odd, $b$ even: odd - even + even = even. Not possible.
Case 4: $a$ even, $b$ even: even - even + even = even.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can't both be even! + Lemma $\implies$ no rational solution.

**Proof of lemma:** Assume a solution of the form $\frac{a}{b}$.

$$\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0$$

Multiply by $b^5$,

$$a^5 - ab^4 + b^5 = 0$$

Case 1: $a$ odd, $b$ odd: odd - odd +odd = even. Not possible.

Case 2: $a$ even, $b$ odd: even - even +odd = even. Not possible.

Case 3: $a$ odd, $b$ even: odd - even +even = even. Not possible.

Case 4: $a$ even, $b$ even: even - even +even = even. Possible.
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( \frac{a}{b} \).

\[
\left( \frac{a}{b} \right)^5 - \frac{a}{b} + 1 = 0
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Multiply by \( b^5 \),

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\]

Case 1: \( a \) odd, \( b \) odd: odd - odd + odd = even. Not possible.
Case 2: \( a \) even, \( b \) odd: even - even + odd = even. Not possible.
Case 3: \( a \) odd, \( b \) even: odd - even + even = even. Not possible.
Case 4: \( a \) even, \( b \) even: even - even + even = even. Possible.

The fourth case is the only one possible,
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( a/b \).

\[
\left( \frac{a}{b} \right)^5 - \frac{a}{b} + 1 = 0
\]

Multiply by \( b^5 \),

\[
a^5 - ab^4 + b^5 = 0
\]

Case 1: \( a \) odd, \( b \) odd: odd - odd +odd = even. Not possible.
Case 2: \( a \) even, \( b \) odd: even - even +odd = even. Not possible.
Case 3: \( a \) odd, \( b \) even: odd - even +even = even. Not possible.
Case 4: \( a \) even, \( b \) even: even - even +even = even. Possible.

The fourth case is the only one possible, so the lemma follows.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.
Proof by cases.

**Theorem:** There exist irrational \( x \) and \( y \) such that \( x^y \) is rational.

Let \( x = y = \sqrt{2} \).

Case 1: \( x^y = \sqrt{2}^\sqrt{2} \) is rational.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2^{\sqrt{2}}}$ is rational. Done!

Case 2: $\sqrt{2^{\sqrt{2}}}$ is irrational.

- New values: $x = \sqrt{2^{\sqrt{2}}}$, $y = \sqrt{2}$. 

Question: Which case holds? 

Don't know!!
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.  
  
  $x^y = \sqrt{2}^{\sqrt{2}}$.
Proof by cases.

**Theorem:** There exist irrational \( x \) and \( y \) such that \( x^y \) is rational.

Let \( x = y = \sqrt{2} \).

Case 1: \( x^y = \sqrt{2} \cdot \sqrt{2} \) is rational. Done!

Case 2: \( \sqrt{2} \cdot \sqrt{2} \) is irrational.

- New values: \( x = \sqrt{2} \cdot \sqrt{2}, \ y = \sqrt{2} \).

Thus, we have irrational \( x \) and \( y \) with a rational \( x^y \) (i.e., 2).

Question: Which case holds?

Don't know!!!
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}, \ y = \sqrt{2}$.
- $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}}$
Proof by cases.

**Theorem:** There exist irrational \( x \) and \( y \) such that \( x^y \) is rational.

Let \( x = y = \sqrt{2} \).

Case 1: \( x^y = \sqrt{2}^{\sqrt{2}} \) is rational. Done!

Case 2: \( \sqrt{2}^{\sqrt{2}} \) is irrational.

- New values: \( x = \sqrt{2}^{\sqrt{2}} \), \( y = \sqrt{2} \).

\[
x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2
\]
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^\sqrt{2}$ is rational. Done!

Case 2: $\sqrt{2}^\sqrt{2}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^\sqrt{2} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2^2} = 2.$$
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

  
  
  \[ x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2^2} = 2. \]

Thus, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

Thus, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).

One of the cases is true so theorem holds.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

▶ New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

▶

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2.$$  

Thus, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2). One of the cases is true so theorem holds. □
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

- 
  \[
  x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2^2} = 2.
  \]

Thus, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).

One of the cases is true so theorem holds.

Question: Which case holds?
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

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- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2^2} = 2.$$ 

Thus, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).

One of the cases is true so theorem holds.

Question: Which case holds? Don’t know!!!
Theorem: $3 = 4$
Be careful.

**Theorem:** \(3 = 4\)

**Proof:** Assume \(3 = 4\).
Be careful.

**Theorem:** 3 = 4

**Proof:** Assume 3 = 4.
Start with 12 = 12.
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get $4 = 3$. 
Theorem: $3 = 4$

Proof: Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get $4 = 3$.

By commutativity
Theorem: $3 = 4$

Proof: Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get $4 = 3$.

By commutativity theorem holds.
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get $4 = 3$.

By commutativity theorem holds.
Theorem: $3 = 4$

Proof: Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get $4 = 3$.

By commutativity theorem holds.

Don’t assume what you want to prove!
Be really careful!

Theorem: $1 = 2$

Proof:
Be really careful!

**Theorem:** \( 1 = 2 \)

**Proof:** For \( x = y \), we have

\[
(x^2 - xy) = x^2 - y^2 = x(x-y) = (x+y)(x-y)
\]

\( x = (x+y)(x-y) \)

Also: Multiplying inequalities by a negative does not mean \( Q \Rightarrow P \Rightarrow Q \).
Be really careful!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$
Be really careful!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

\[(x^2 - xy) = x^2 - y^2\]
\[x(x - y) = (x + y)(x - y)\]

Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

$P \implies Q$ does not mean $Q \implies P$. 
Be really careful!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

\[
(x^2 - xy) = x^2 - y^2
\]

\[
x(x - y) = (x + y)(x - y)
\]

\[
x = (x + y)
\]
Be really careful!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

$(x^2 - xy) = x^2 - y^2$

$x(x - y) = (x + y)(x - y)$

$x = (x + y)$

$x = 2x$
Theorem: $1 = 2$
Proof: For $x = y$, we have

\[(x^2 - xy) = x^2 - y^2\]
\[x(x - y) = (x + y)(x - y)\]
\[x = (x + y)\]
\[x = 2x\]
\[1 = 2\]
Theorem: \(1 = 2\)
Proof: For \(x = y\), we have

\[
(x^2 - xy) = x^2 - y^2 \\
x(x - y) = (x + y)(x - y) \\
x = (x + y) \\
x = 2x \\
1 = 2
\]
Theorem: $1 = 2$
Proof: For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$

$$x(x - y) = (x + y)(x - y)$$

$$x = (x + y)$$

$$x = 2x$$

$$1 = 2$$

Dividing by zero is no good.
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**Theorem:** \(1 = 2\)

**Proof:** For \(x = y\), we have

\[
(x^2 - xy) = x^2 - y^2
\]

\[
x(x - y) = (x + y)(x - y)
\]

\[
x = (x + y)
\]

\[
x = 2x
\]

\[
1 = 2
\]

Dividing by zero is no good.

Also: Multiplying inequalities by a negative.
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**Theorem:** \( 1 = 2 \)

**Proof:** For \( x = y \), we have

\[
(x^2 - xy) = x^2 - y^2
\]
\[
x(x - y) = (x + y)(x - y)
\]
\[
x = (x + y)
\]
\[
x = 2x
\]

\[
1 = 2
\]

Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

\( P \implies Q \) does not mean \( Q \implies P \).
Summary: Note 2.

Direct Proof:

...
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. 

By Contraposition:
To Prove: $P \implies Q$
Assume $\neg Q$.
Prove $\neg P$.

By Contradiction:
To Prove: $P$
Assume $\neg P$.
Prove False.

By Cases:
Universal: show that statement holds in all cases.
Existence: used cases where one is true.

Either $\sqrt{2}$ and $\sqrt{2}$ worked.
or $\sqrt{2}$ and $\sqrt{2}$ worked.

Careful when proving!
Don't assume the theorem.
Divide by zero.
Watch converse.


Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \).
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$. Assume $\neg Q$.

By Contradiction:

By Cases: informal.
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To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

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To Prove: $P$
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \).
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.
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To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \). Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \). Assume \( \neg P \). Prove \( \text{False} \).

By Cases: informal.
Universal: show that statement holds in all cases.
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
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Either $\sqrt{2}$ and $\sqrt{2}$ worked.
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Careful when proving!
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Divide by zero.
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Either $\sqrt{2}$ and $\sqrt{2}$ worked.

Careful when proving!
Don’t assume the theorem.
Divide by zero.
Watch converse.

...
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
   Either $\sqrt{2}$ and $\sqrt{2}$ worked.
   or $\sqrt{2}$ and $\sqrt{2^{\sqrt{2}}}$ worked.

Careful when proving!
Don’t assume the theorem.
Divide by zero.
Watch converse.
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove \text{False}.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.

Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
   or \( \sqrt{2} \) and \( \sqrt{2}^{\sqrt{2}} \) worked.

Careful when proving!
Don't assume the theorem.
Divide by zero.
Watch converse.
...
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
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Existence: used cases where one is true.
   Either $\sqrt{2}$ and $\sqrt{2}$ worked.
   or $\sqrt{2}$ and $\sqrt{2} \sqrt{2}$ worked.

Careful when proving!
Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
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   Either $\sqrt{2}$ and $\sqrt{2}$ worked.
   or $\sqrt{2}$ and $\sqrt{2}^{\sqrt{2}}$ worked.

Careful when proving!
Don’t assume the theorem.
Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
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 Either $\sqrt{2}$ and $\sqrt{2}$ worked.
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To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

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Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
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Careful when proving!
Don’t assume the theorem. Divide by zero. Watch converse.
Summary: Note 2.

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove False.

By Cases: informal.
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   Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
   or \( \sqrt{2} \) and \( \sqrt{2} \) worked.

Careful when proving!
Don’t assume the theorem. Divide by zero. Watch converse. ...
Poll.
Poll.

1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.
The natural numbers.
The natural numbers.
The natural numbers.
The natural numbers.
The natural numbers.

0, 1, 2,
The natural numbers.

0, 1, 2, 3,
The natural numbers.

0, 1, 2, 3, ...

...
The natural numbers.

0, 1, 2, 3, ..., n,
The natural numbers.

0, 1, 2, 3, 
... , n, n+1,
The natural numbers.
The natural numbers.

$0, 1, 2, 3, \ldots, n, n+1, n+2, n+3, \ldots$
A formula.

Teacher: Hello class.

Teacher: Please add the numbers from 1 to 100.

Gauss: It's \((100)(101)\) or 5050!

Five year old Gauss Theorem:

\[ \forall (n \in \mathbb{N}) : \sum n_i = 0 \iff (n)(n+1)\] 2.

It is a statement about all natural numbers.

\[ \forall (n \in \mathbb{N}) : P(n). \]

\[ P(n) \] is "\[ \sum n_i = 0 \iff (n)(n+1)\] 2."

Principle of Induction:

\[ \text{Prove } P(0). \]

\[ \text{Assume } P(k), \text{ "Induction Hypothesis" } \]

\[ \text{Prove } P(k+1). \text{ "Induction Step." } \]
A formula.

Teacher: Hello class.
A formula.

Teacher: Hello class.
Teacher:
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It's

A formula.
Teacher: Hello class.
Teacher: *Please add the numbers from 1 to 100.*
Gauss: It’s $\frac{(100)(101)}{2}$
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It's \[ \frac{(100)(101)}{2} \] or 5050!

Five year old Gauss Theorem: \( \forall (n \in \mathbb{N}) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \).
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It’s \( \frac{(100)(101)}{2} \) or 5050!

Five year old Gauss Theorem: \( \forall (n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \).
It is a statement about all natural numbers.
Teacher: Hello class.
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Five year old Gauss Theorem: \( \forall (n \in \mathbb{N}) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \).

It is a statement about all natural numbers.
\( \forall (n \in \mathbb{N}) : P(n) \).
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It’s \( \frac{(100)(101)}{2} \) or 5050!

Five year old Gauss Theorem: \( \forall (n \in \mathbb{N}) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}. \)

It is a statement about all natural numbers.

\( \forall (n \in \mathbb{N}) : P(n). \)

\( P(n) \) is “\( \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \).”
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It’s \( \frac{(100)(101)}{2} \) or 5050!

Five year old Gauss Theorem: \( \forall (n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \).
It is a statement about all natural numbers.

\( \forall (n \in N) : P(n) \).

\( P(n) \) is “\( \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \)”.

Principle of Induction:

▶ Prove \( P(0) \).
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It’s \( \frac{(100)(101)}{2} \) or 5050!

Five year old Gauss Theorem: \( \forall (n \in \mathbb{N}) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \).
It is a statement about all natural numbers.
\[ \forall (n \in \mathbb{N}) : P(n). \]
\( P(n) \) is “\( \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \)”.

Principle of Induction:
- Prove \( P(0) \).
- Assume \( P(k) \), “Induction Hypothesis”
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- Prove \( P(0) \).
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- Prove \( P(k + 1) \). “Induction Step.”
Gauss induction proof.

**Theorem:** For all natural numbers $n$, $0 + 1 + 2 \cdots n = \frac{n(n+1)}{2}$
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$P(k+1)!$ By principle of induction...
Notes visualization

Note’s visualization: an infinite sequence of dominoes.

Prove they all fall down;
Note’s visualization: an infinite sequence of dominoes.

Prove they all fall down;

- $P(0) = \text{“First domino falls”}$
Note’s visualization: an infinite sequence of dominos.

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- $(\forall k) \ P(k) \implies P(k + 1)$:
Note’s visualization: an infinite sequence of dominos.

Prove they all fall down;

- $P(0) = \text{“First domino falls”}$
- $(\forall k) \ P(k) \implies P(k + 1): \ “k\text{th domino falls implies that }k + 1\text{st domino falls}”$
Climb an infinite ladder?

\[ P(n) \Rightarrow P(n+1) \]

\[ \forall k, P(k) = \Rightarrow P(k+1) \]

\[ P(0) = \Rightarrow P(1) = \Rightarrow P(2) = \Rightarrow P(3) = \ldots \]

Your favorite example of forever..
or the natural numbers...
Climb an infinite ladder?
Climb an infinite ladder?

∀ k, P(k) =⇒ P(k + 1)

∀ n ∈ N, P(n)

Your favorite example of forever... or the natural numbers...

P(0)
Climb an infinite ladder?

\[ P(0) \]
\[ P(1) \]
\[ P(2) \]
\[ P(3) \]
\[ P(n) \]
\[ P(n+1) \]
\[ P(n+2) \]
\[ P(n+3) \]

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Climb an infinite ladder?

∀ₖ, P(ₖ) ⇒ P(ₖ + 1)

P(0) ⇒ P(1) ⇒ P(2) ⇒ P(3) …
Climb an infinite ladder?

\[ P(n) \]

\[ \forall k, P(k) \implies P(k+1) \]

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∀\(k\), \(P(k) \implies P(k+1)\)

\(P(0) \implies P(1) \implies P(2) \implies P(3) \ldots\)
Climb an infinite ladder?

\[ P(0) \]
\[ P(1) \]
\[ P(2) \]
\[ P(3) \]
\[ \vdots \]
\[ P(n) \] 
\[ P(n+1) \] 
\[ P(n+2) \] 
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Your favorite example of forever..or the natural numbers...
Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Proof?

Idea: assume predicate \(P(n)\) for \(n = k\).

\(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate, \(P(n)\) true for \(n = k + 1\)?

\(\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}\).

How about \(k+2\). Same argument starting at \(k+1\) works!

Induction Step.

\(P(k) = \Rightarrow P(k+1)\).

Is this a proof?

It shows that we can always move to the next step.

Need to start somewhere.

\(P(0)\) is \(\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}\) Base Case.

Statement is true for \(n = 0\)

\(\neg P(0)\) is true plus inductive step = \(\Rightarrow\) true for \(n = 1\)

\((P(0) \land (P(0) = \Rightarrow P(1)))) = \Rightarrow P(1)\) plus inductive step = \(\Rightarrow\) true for \(n = 2\)

... true for \(n = k\) = \(\Rightarrow\) true for \(n = k+1\)

... Predicate, \(P(n)\), True for all natural numbers!

Proof by Induction.
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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\[\sum_{i=1}^{k+1} i\]
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How about \(k + 2\).
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Statement is true for \(n = 0\)
Gauss and Induction

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Statement is true for \(n = 0\) \(P(0)\) is true.
Gauss and Induction

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plus inductive step
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate \(P(n)\) for \(n = k\). \(P(k)\) is \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate, \(P(n)\) true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.** \(P(k) \implies P(k + 1)\).

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(P(0)\) is \(\sum_{i=0}^{0} i = 0 = \frac{(0)(0+1)}{2}\) **Base Case.**

Statement is true for \(n = 0\) \(P(0)\) is true

plus inductive step \(\implies\) true for \(n = 1\)
Gauss and Induction

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Statement is true for \(n = 0\) \(P(0)\) is true

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Gauss and Induction

Child Gauss: $(\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k + 1$?

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Statement is true for $n = 0$ $P(0)$ is true 

plus inductive step $\implies$ true for $n = 1$ $(P(0) \land (P(0) \implies P(1))) \implies P(1)$ 

plus inductive step
Gauss and Induction

Child Gauss: \( (\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2} ) \) Proof?

Idea: assume predicate \( P(n) \) for \( n = k \). \( P(k) \) is \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \).

Is predicate, \( P(n) \) true for \( n = k + 1 \)?

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Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \( P(0) \) is \( \sum_{i=0}^{0} i = 0 = \frac{(0)(0+1)}{2} \) **Base Case.**

Statement is true for \( n = 0 \) \( P(0) \) is true

\[ \text{plus inductive step } \implies \text{true for } n = 1 \quad (P(0) \land (P(0) \implies P(1))) \implies P(1) \]

\[ \text{plus inductive step } \implies \text{true for } n = 2 \]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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\(\) plus inductive step \(\implies\) true for \(n = 1\) \((P(0) \land (P(0) \implies P(1))) \implies P(1)\)

\(\) plus inductive step \(\implies\) true for \(n = 2\) \((P(1) \land (P(1) \implies P(2))) \implies P(2)\)
Gauss and Induction

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Gauss and Induction

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plus inductive step \(\implies\) true for \(n = 1\) \(P(0) \land (P(0) \implies P(1)) \implies P(1)\)

plus inductive step \(\implies\) true for \(n = 2\) \(P(1) \land (P(1) \implies P(2)) \implies P(2)\)

\[\ldots\]

true for \(n = k\)
Gauss and Induction

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\[
\vdots
\]

\[
\text{true for } n = k \implies \text{true for } n = k + 1 \quad (P(k) \land (P(k) \implies P(k+1))) \implies P(k+1)
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plus inductive step \(\implies\) true for \(n = 2\) \((P(1) \wedge (P(1) \implies P(2))) \implies P(2)\)

... \(\implies\) true for \(n = k\) \(\implies\) true for \(n = k + 1\) \((P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)\)

...

Predicate, \(P(n)\), **True** for all natural numbers!
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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true for \(n = k\) \(\implies\) true for \(n = k + 1\) \((P(k) \land (P(k) \implies P(k+1))) \implies P(k+1)\)

\[
\vdots
\]

Predicate, \(P(n)\), **True** for all natural numbers! **Proof by Induction.**
Induction

The canonical way of proving statements of the form

\((\forall k \in \mathbb{N})(P(k))\)
Induction

The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \( n \), \( 1 + 2 \cdots n = \frac{n(n+1)}{2} \).
The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
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The basic form
Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \( n \), \( 1 + 2 \cdots n = \frac{n(n+1)}{2} \).
- For all \( n \in \mathbb{N} \), \( n^3 - n \) is divisible by 3.
- The sum of the first \( n \) odd integers is a perfect square.

The basic form

- Prove \( P(0) \). “Base Case”.
- \( P(k) \implies P(k + 1) \)
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k + 1)$
  - Assume $P(k)$, “Induction Hypothesis”
Induction

The canonical way of proving statements of the form

$$\forall k \in \mathbb{N}(P(k))$$

- For all natural numbers \( n \), \( 1 + 2 \cdots n = \frac{n(n+1)}{2} \).
- For all \( n \in \mathbb{N} \), \( n^3 - n \) is divisible by 3.
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The basic form

- Prove \( P(0) \). “Base Case”.
- \( P(k) \implies P(k+1) \)
  - Assume \( P(k) \), “Induction Hypothesis”
  - Prove \( P(k+1) \). “Induction Step.”
Induction

The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \( n \), \( 1 + 2 \cdots n = \frac{n(n+1)}{2} \).
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\( P(n) \) true for all natural numbers \( n \)!!!
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

▶ For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
▶ For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.
▶ The sum of the first $n$ odd integers is a perfect square.

The basic form

▶ Prove $P(0)$. “Base Case”.
▶ $P(k) \implies P(k+1)$
  ▶ Assume $P(k)$, “Induction Hypothesis”
  ▶ Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers $n$!!!
Get to use $P(k)$ to prove $P(k+1)$!
Induction

The canonical way of proving statements of the form

\[(\forall k \in N)(P(k))\]

- For all natural numbers \( n \), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
- For all \( n \in N \), \( n^3 - n \) is divisible by 3.
- The sum of the first \( n \) odd integers is a perfect square.

The basic form

- Prove \( P(0) \). “Base Case”.
- \( P(k) \implies P(k+1) \)
  - Assume \( P(k) \), “Induction Hypothesis”
  - Prove \( P(k+1) \). “Induction Step.”

\( P(n) \) true for all natural numbers \( n \)!!!
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- $P(k) \implies P(k+1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers $n$!!!
Get to use $P(k)$ to prove $P(k+1)$!!!
Next Time.

More induction!
Next Time.

More induction!
See you on Thursday!