

## CS70: Lecture 2. Outline.

Today: Proofs!!!

1. By Example.
2. Direct. (Prove  $P \implies Q$ .)
3. by Contraposition (Prove  $P \implies Q$ )
4. by Contradiction (Prove  $P$ .)
5. by Cases

If time: discuss induction.

## Quick Background and Notation.

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A natural number  $p > 1$ , is **prime** if it is divisible only by 1 and itself.

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Therefore  $Q$ .

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$$100a + 10b + c = 11k + 99a + 11b$$

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Don't assume what you want to prove!

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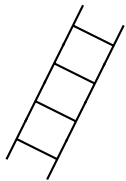
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# CS70: Note 3. Induction!

1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.

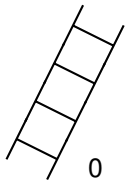
The natural numbers.

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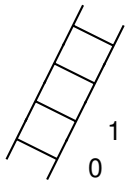
0,





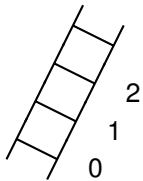
The natural numbers.

0, 1,



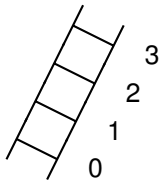
The natural numbers.

0, 1, 2,

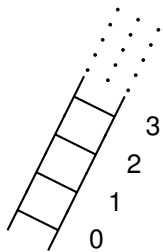


The natural numbers.

0, 1, 2, 3,

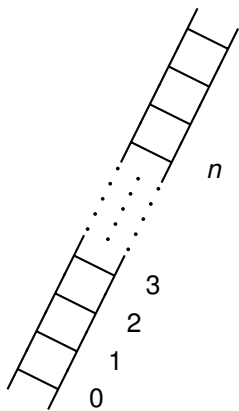


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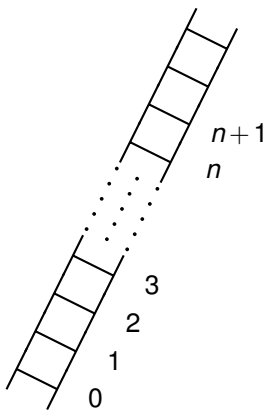
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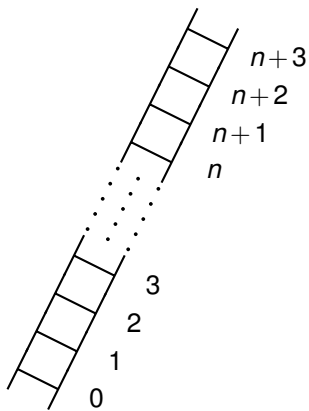
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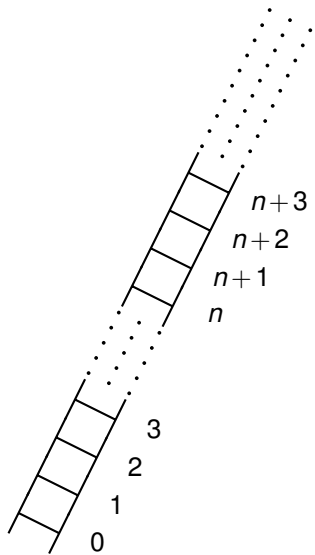
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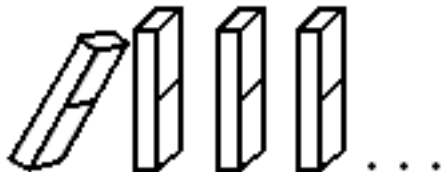
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## Notes visualization

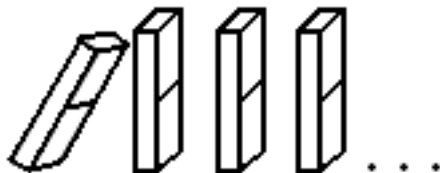
Note's visualization: an infinite sequence of dominos.



Prove they all fall down;

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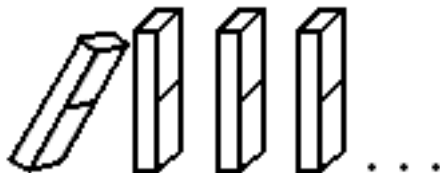


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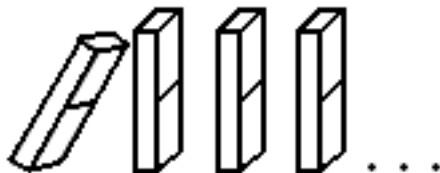


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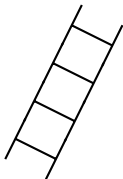
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Climb an infinite ladder?

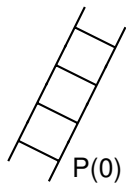


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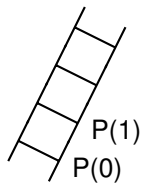


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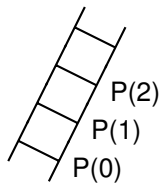


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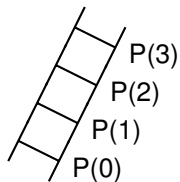
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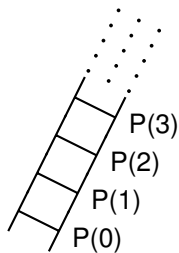
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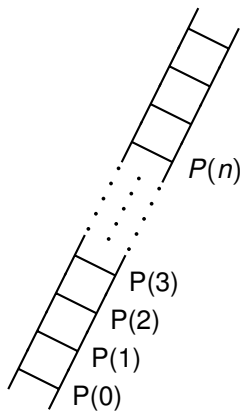
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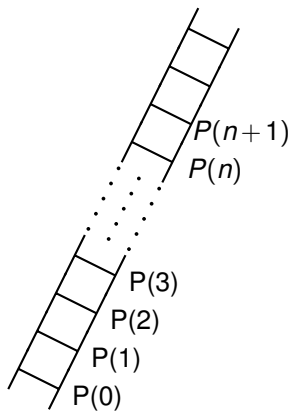
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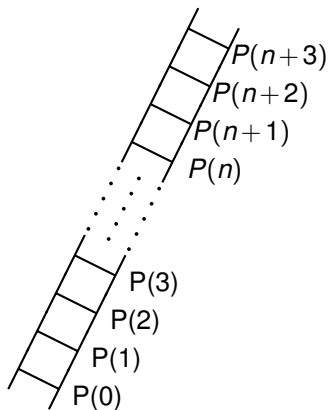
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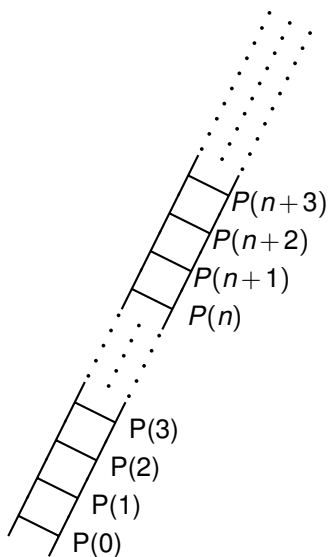


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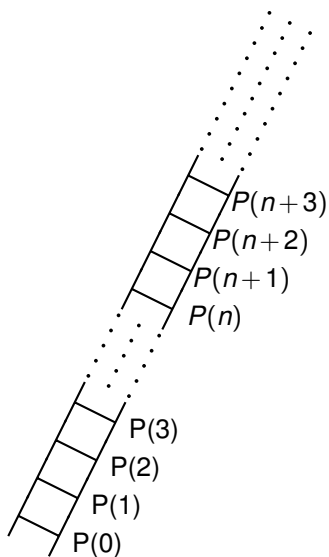
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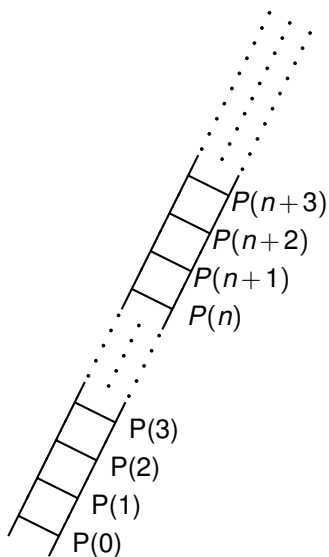
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Your favorite example of forever..or the natural numbers...

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Next Time.

More induction!

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See you on Tuesday!