CS70: Lecture 28.

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Continuous Probability

1. Conditional Probability (Recap: revisit $G(p)$)
2. Continuous Probability: Examples
3. Continuous Probability: Events
4. Continuous Random Variables

Recap: Conditional distributions

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Total Probability Theorem: If $A_1, A_2, \ldots, A_N$ partition Ω, and $P[A_i] > 0 \; \forall i$, then

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Let's visit the mean and variance of the geometric distribution using conditional expectation.

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&= E[X+1]
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Derive the variance for $X \sim G(p)$ by finding $E[X^2]$ using conditioning.

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$\Rightarrow$ $2H$ will essentially NEVER appear!

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$$P[Y = k] = P[Y = k \mid A]P[A] + P[Y = k \mid D]P[D]$$

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$P[D] \to 0$ as n grows

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Limit: "$Poisson(\lambda_1) + Poisson(\lambda_2) = Poisson(\lambda_1 + \lambda_2)$"

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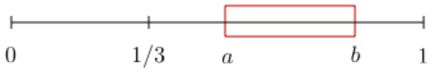

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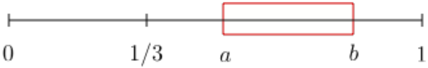

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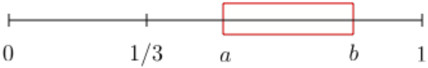

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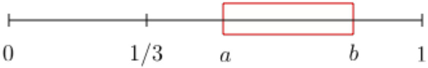

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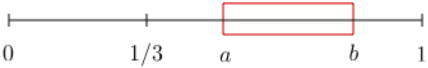

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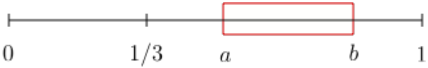

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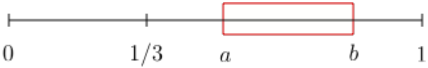

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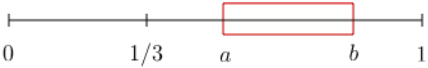

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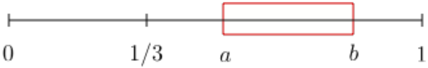

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Makes sense: $b - a$ is the fraction of $[0,1]$ that $[a,b]$ covers.

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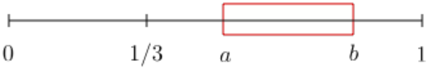

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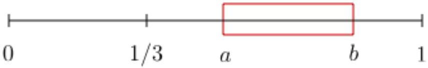

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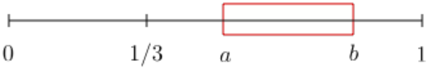

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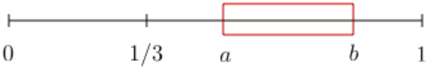

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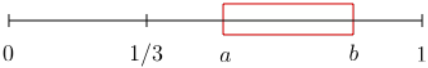

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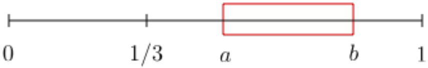

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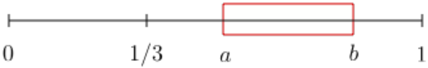

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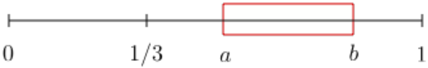

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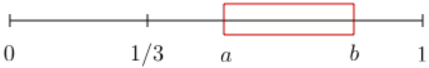

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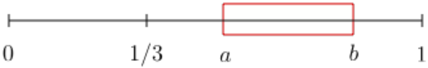

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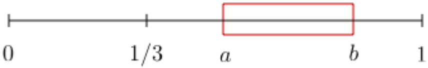

Note: A **radical** change in approach. For a finite probability space, $\Omega = \{1, 2, \ldots, N\}$, we started with $Pr[\omega] = p_\omega$. We then defined $Pr[A] = \sum_{\omega \in A} p_\omega$ for $A \subset \Omega$. We used the same approach for countable Ω.

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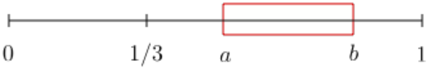

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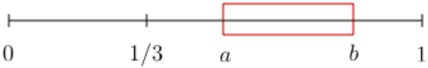
Uniformly at Random in $[0,1]$.



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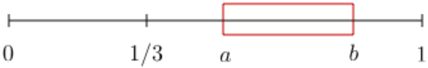

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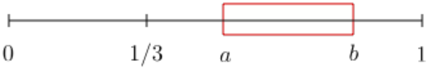

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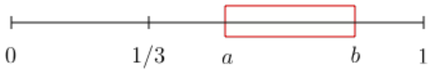

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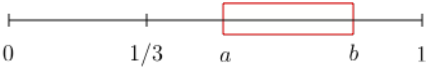

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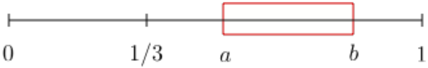

Note: $Pr[X \leq x] = x$ for $x \in [0,1]$.

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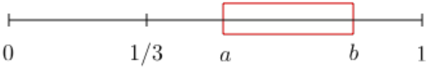

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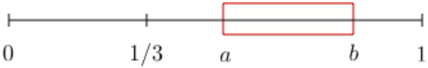

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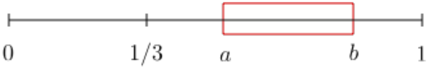

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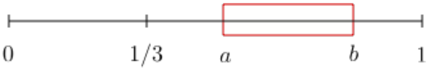

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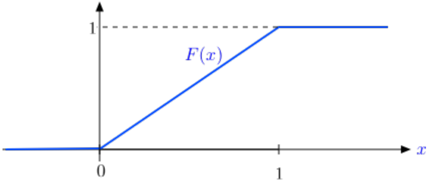

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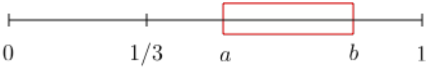

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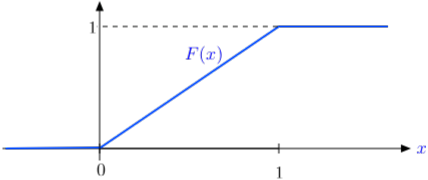
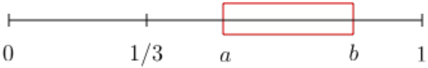


Then we have $Pr[X \in (a, b]] = Pr[X \leq b] - Pr[X \leq a]$

Uniformly at Random in $[0, 1]$.



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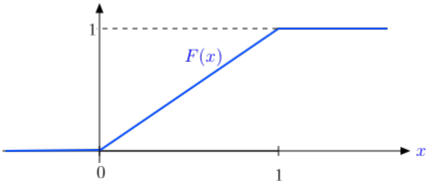

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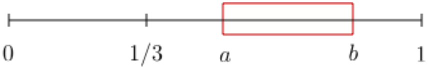

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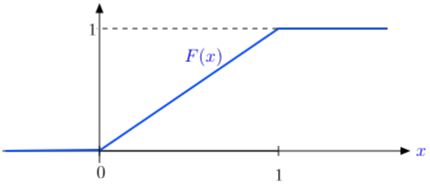
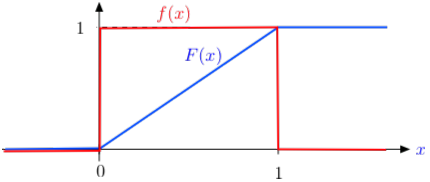


Then we have $Pr[X \in (a, b]] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a)$.

Thus, $F(\cdot)$ specifies the probability of all the events!

Uniformly at Random in $[0,1]$.



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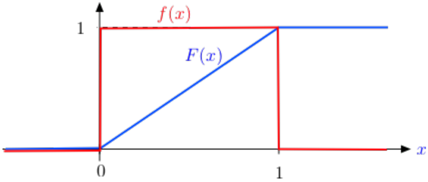

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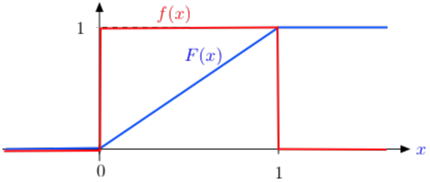

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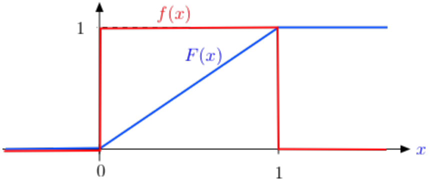

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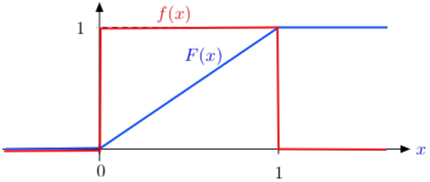

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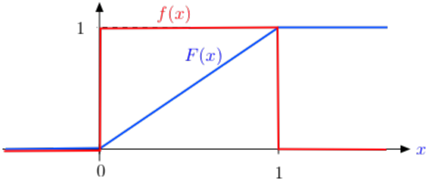

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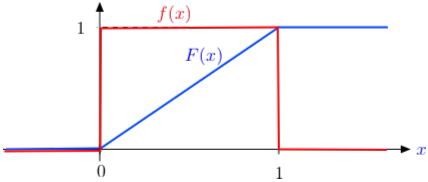
Thus, the probability of an event is the integral of $f(x)$ over the event:

Uniformly at Random in $[0, 1]$.



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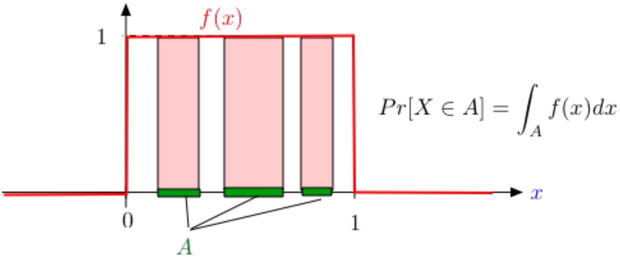
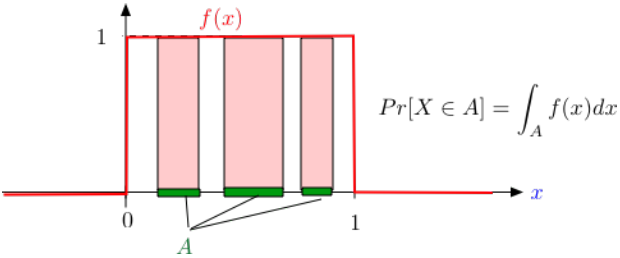
Thus, the probability of an event is the integral of $f(x)$ over the event:

$$Pr[X \in A] = \int_A f(x)dx.$$

Uniformly at Random in $[0, 1]$.

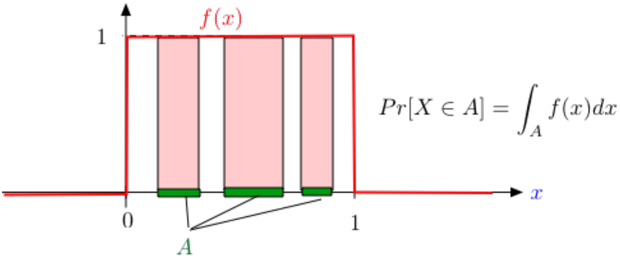


Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
one unit of probability is spread over $[0, 1]$:

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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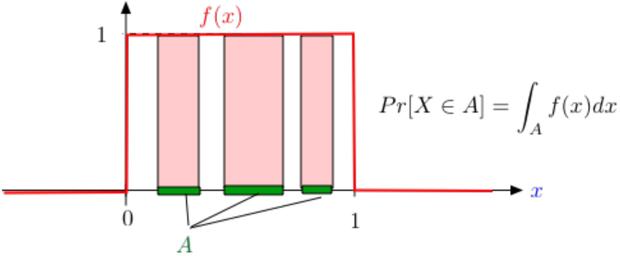

Think of $f(x)$ as describing how
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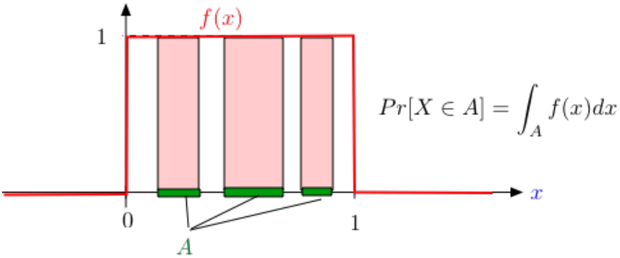
Then $Pr[X \in A]$ is the probability mass over A.

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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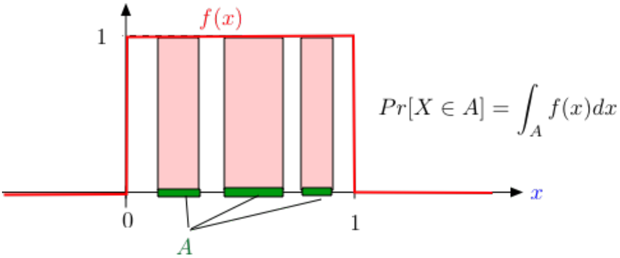
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
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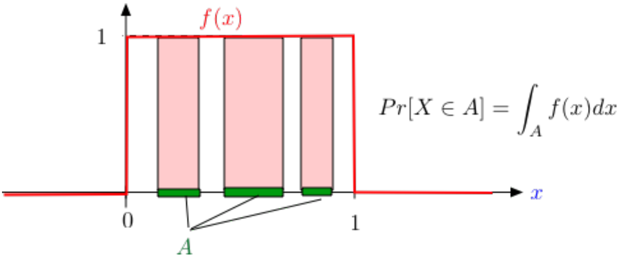
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

- This makes the probability automatically additive.

Uniformly at Random in $[0, 1]$.



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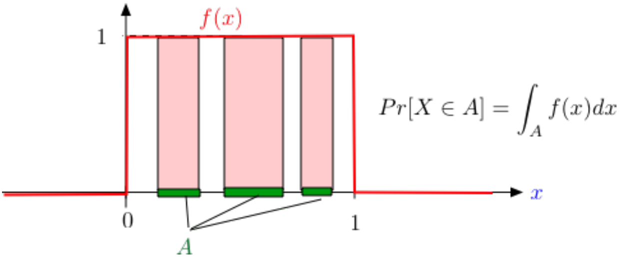
Uniformly at Random in $[0, 1]$.



Think of $f(x)$ as describing how
one unit of probability is spread over $[0, 1]$: uniformly!

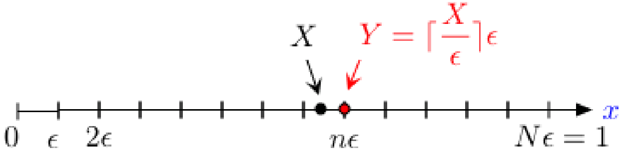
Then $Pr[X \in A]$ is the probability mass over A.

Observe:

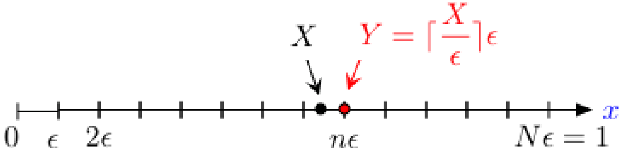
- This makes the probability automatically additive.
- We need $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Uniformly at Random in $[0, 1]$.

Uniformly at Random in $[0, 1]$.

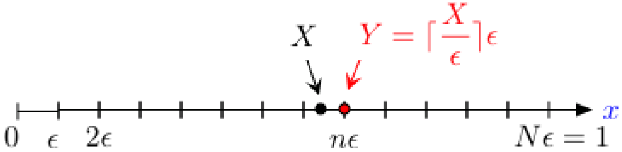


Uniformly at Random in $[0, 1]$.



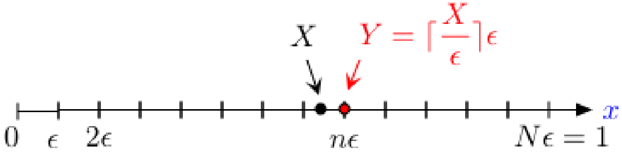
Discrete Approximation:

Uniformly at Random in $[0, 1]$.



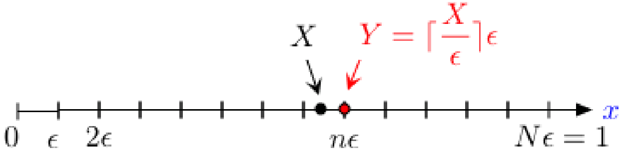
Discrete Approximation: Fix $N \gg 1$

Uniformly at Random in $[0, 1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

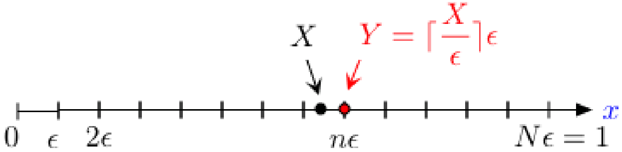
Uniformly at Random in $[0,1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Uniformly at Random in $[0,1]$.



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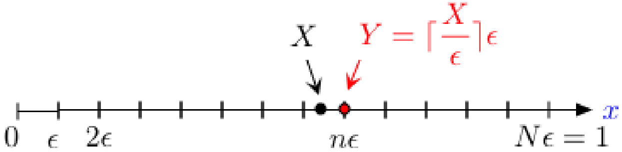
Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1,\ldots,N$.

Then $|X - Y| \leq \varepsilon$

Uniformly at Random in $[0,1]$.

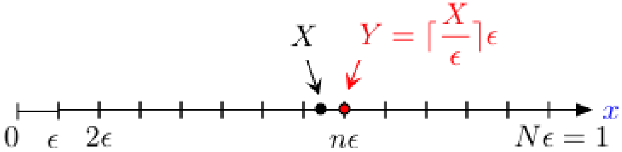


Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \le n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \le \varepsilon$ and Y is discrete:

Uniformly at Random in $[0,1]$.



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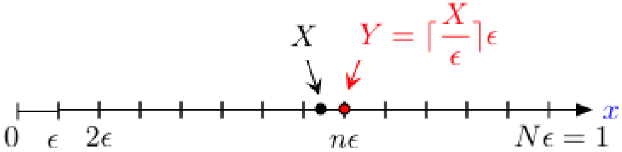
Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Uniformly at Random in $[0,1]$.



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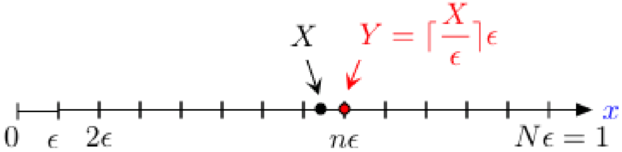
Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

Uniformly at Random in $[0,1]$.



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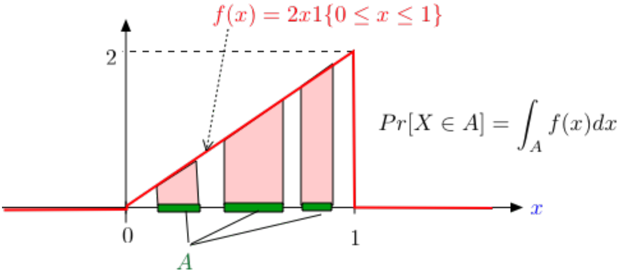
Then $|X - Y| \le \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

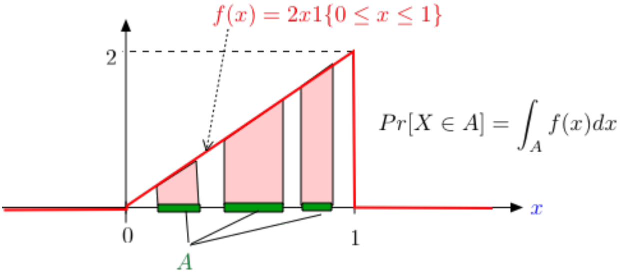
Thus, X is 'almost discrete.'

Nonuniformly at Random in $[0, 1]$.

Nonuniformly at Random in $[0, 1]$.



Nonuniformly at Random in $[0, 1]$.



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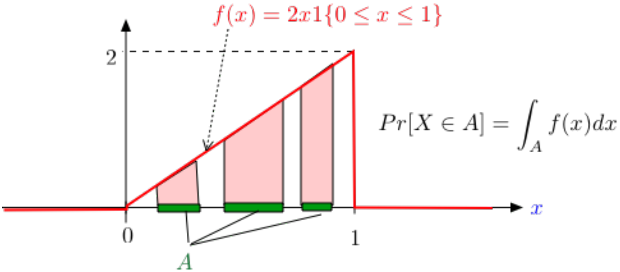

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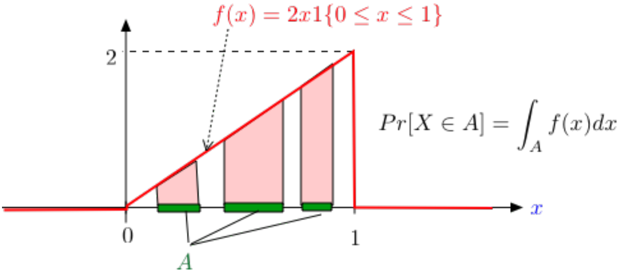

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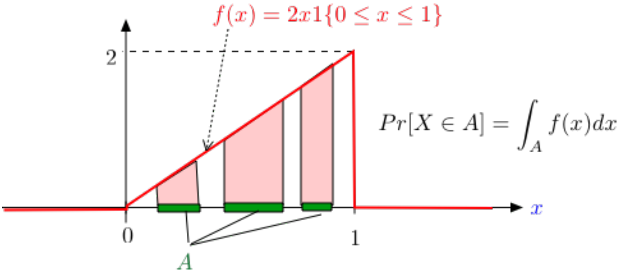

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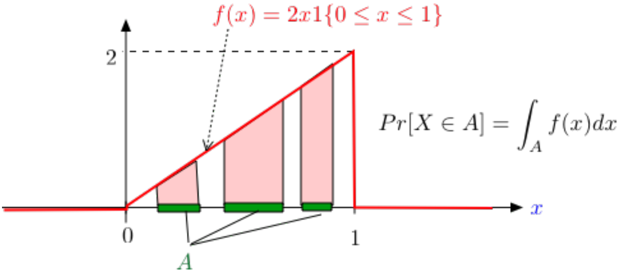

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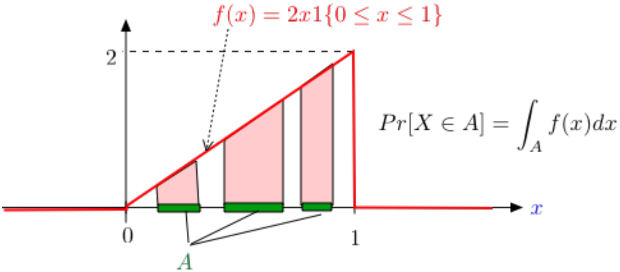

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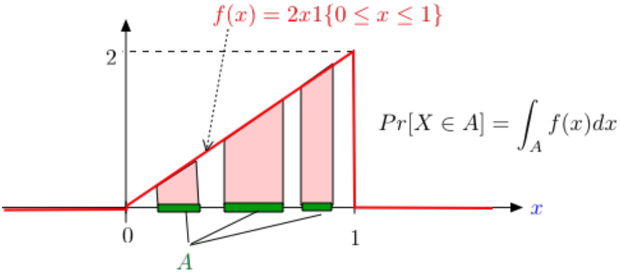

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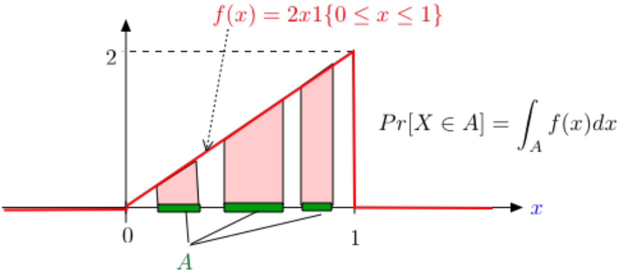

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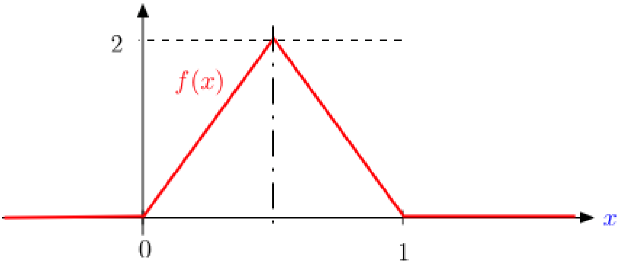

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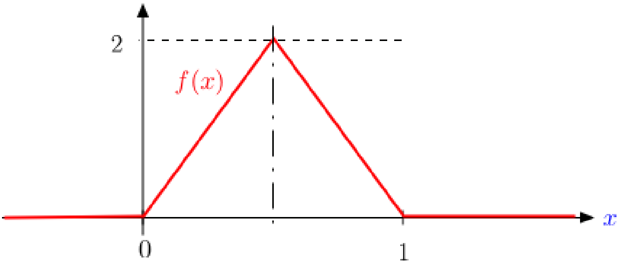

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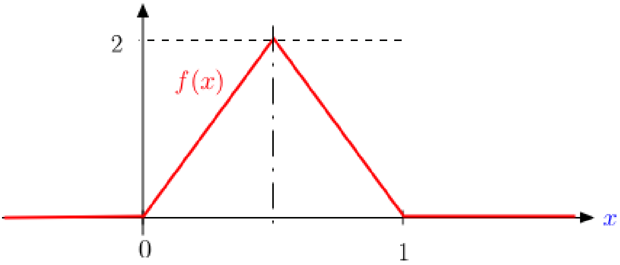

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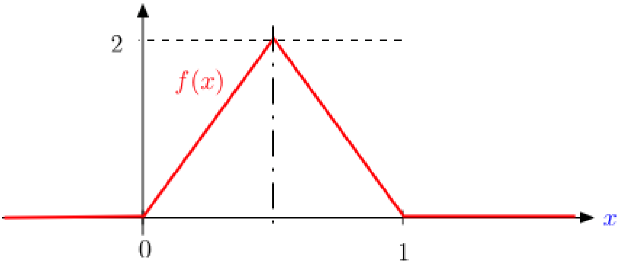

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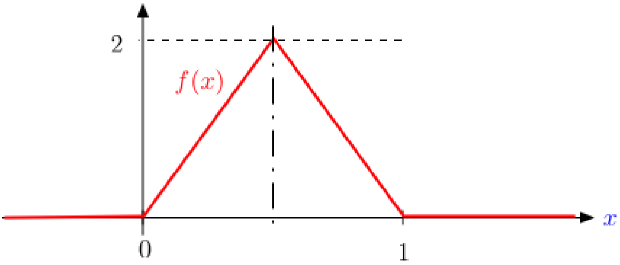

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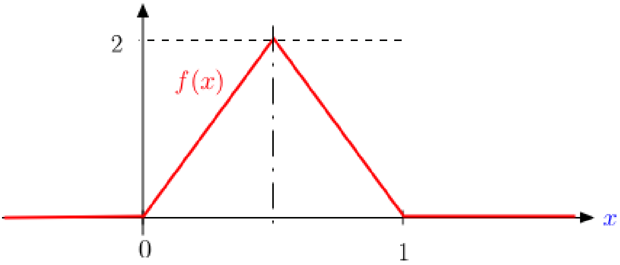

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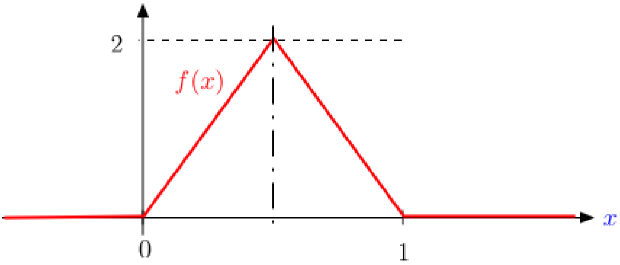

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For instance, $Pr[X \in [0,1/3]] = \int_0^{1/3} 4x dx = 2\left[x^2\right]_0^{1/3} = \frac{2}{9}$.

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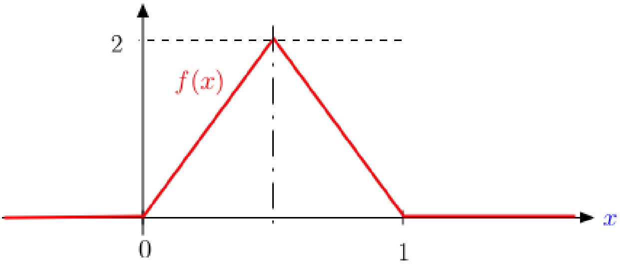

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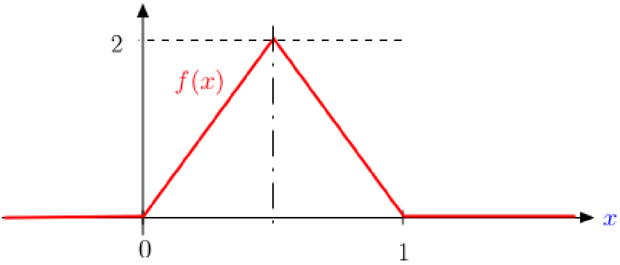

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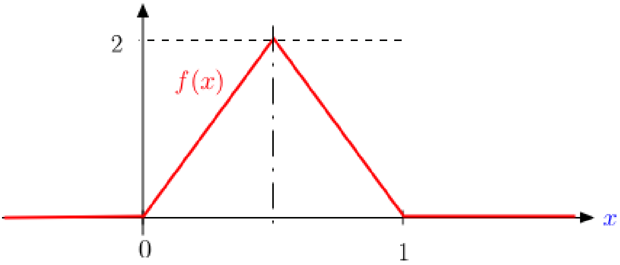

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General Random Choice in $\mathfrak{R}$

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An illustration of $Pr[X \in (x, x+\varepsilon)] \approx f_X(x)\varepsilon$:

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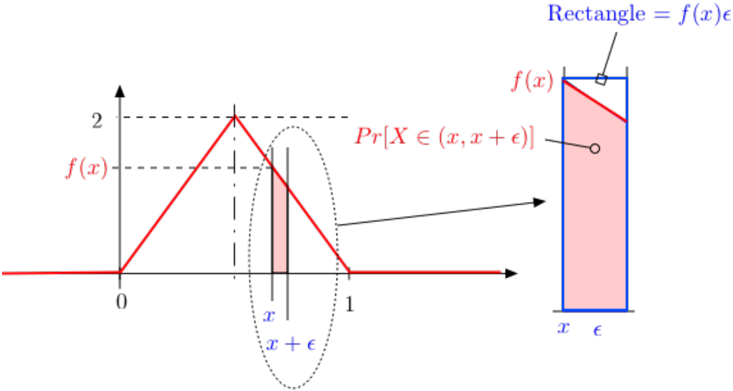

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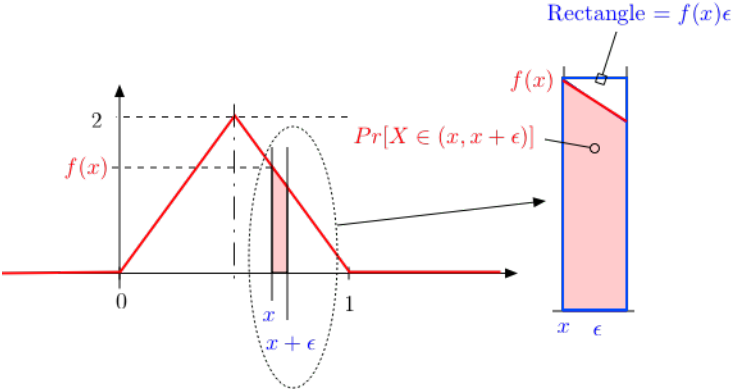

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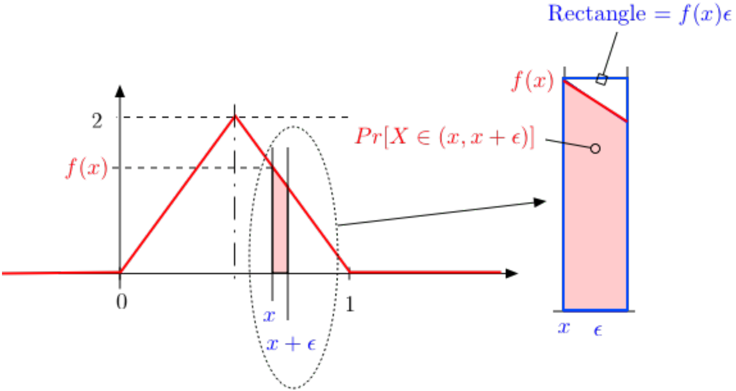

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Also, if $f_X(x) = \frac{d}{dx}F_X(x)$, then $F_X(x+\varepsilon) - F_X(x) \approx f_X(x)\varepsilon$.

Hence, $Pr[Y = n\varepsilon] \approx f_X(n\varepsilon)\varepsilon$.

Thus, we can think of X of being almost discrete with $Pr[X = n\varepsilon] \approx f_X(n\varepsilon)\varepsilon$.

Discrete Approximation

Fix $\varepsilon \ll 1$ and let $Y = n\varepsilon$ if $X \in (n\varepsilon, (n+1)\varepsilon]$.

Thus, $Pr[Y = n\varepsilon] = F_X((n+1)\varepsilon) - F_X(n\varepsilon)$.

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Example: CDF

Example: hitting random location on gas tank.

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Random location on circle.

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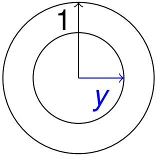

Random Variable: Y distance from center.

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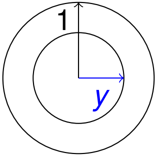

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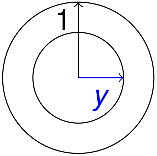

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$$Pr[Y \leq y] = \frac{\text{area of small circle}}{\text{area of dartboard}}$$

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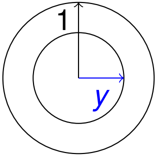

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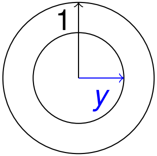

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Hence,

$$F_Y(y) = Pr[Y \leq y] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Calculation of event with dartboard..

Probability between .5 and .6 of center?

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PDF.

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The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.

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The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.
Use whichever is convenient.

Target

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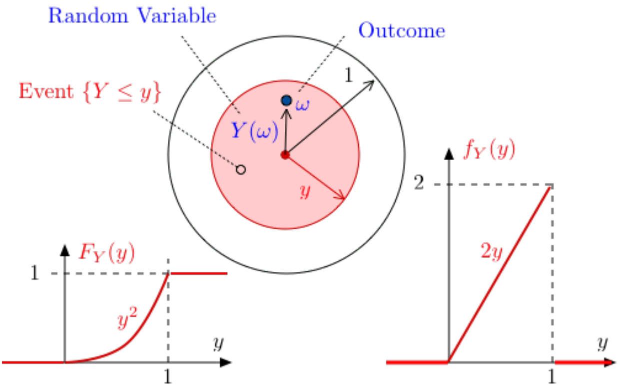
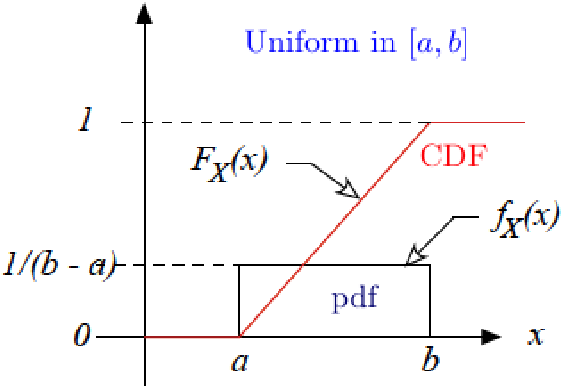


Random Variable
Outcome
Event $\{Y \leq y\}$
1
ω
$Y(\omega)$
y
$f_Y(y)$
2
$2y$
$F_Y(y)$
1
y^2
y
1
y
1

$U[a, b]$

$U[a,b]$



Expo(λ)

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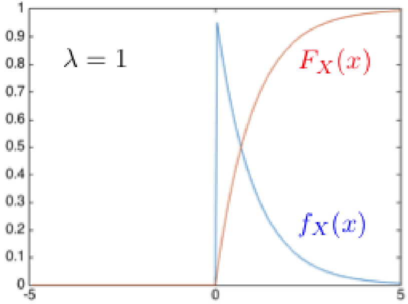

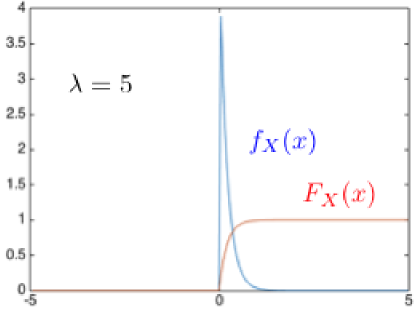

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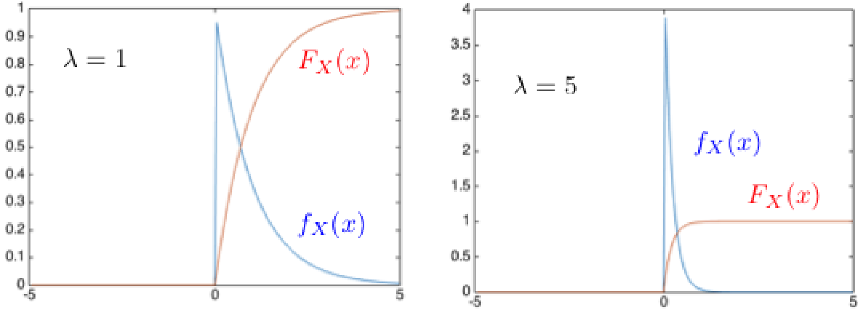

Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

Random Variables

Continuous random variable X, specified by

1. $F_X(x) = Pr[X \leq x]$ for all x.

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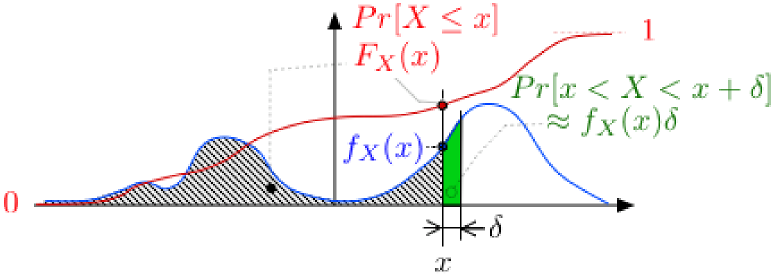
Recall that $Pr[X \in (x, x+\delta)] \approx f_X(x)\delta$. Think of X taking discrete values $n\delta$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$ with $Pr[X = n\delta] = f_X(n\delta)\delta$.

A Picture



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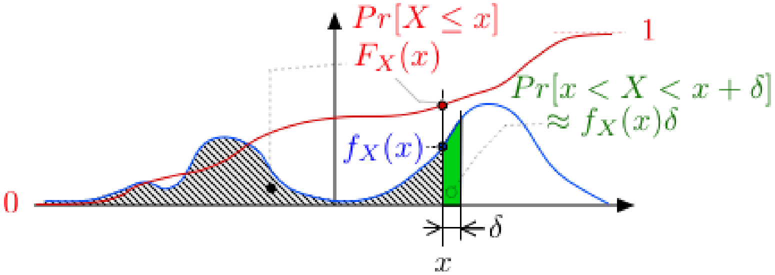

The pdf $f_X(x)$ is a nonnegative function that integrates to 1.

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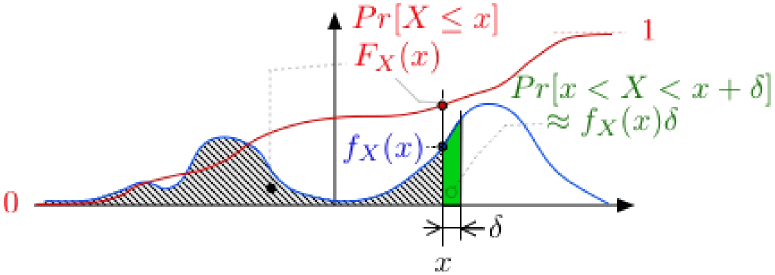

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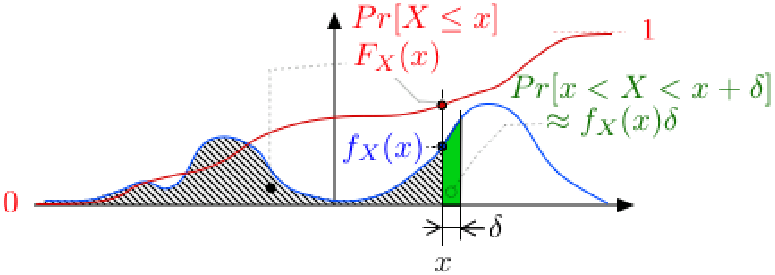

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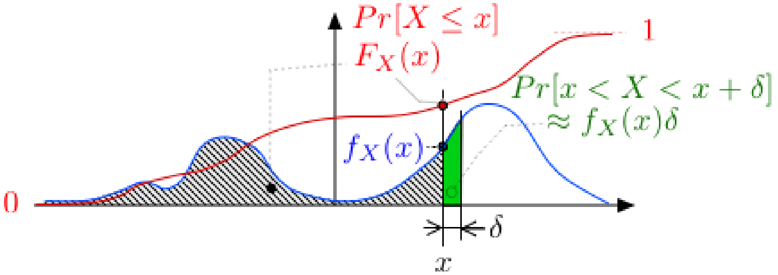

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Summary

Continuous Probability

1. pdf:

Summary

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Summary

Continuous Probability

1. pdf: $Pr[X \in (x, x+\delta]] = f_X(x)\delta$.
2. CDF: $Pr[X \leq x] = F_X(x) = \int_{-\infty}^{x} f_X(y)dy$.
3. *U*[*a*, *b*]: $f_X(x) = \frac{1}{b-a}1\{a \leq x \leq b\}; F_X(x) = \frac{x-a}{b-a}$ for $a \leq x \leq b$.
4. *Expo*(λ):
   $f_X(x) = \lambda \exp\{-\lambda x\}1\{x \geq 0\}; F_X(x) = 1 - \exp\{-\lambda x\}$ for $x \leq 0$.
5. Target: $f_X(x) = 2x1\{0 \leq x \leq 1\}$;

Summary

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3. $U[a,b]$: $f_X(x) = \frac{1}{b-a}1\{a \leq x \leq b\}; F_X(x) = \frac{x-a}{b-a}$ for $a \leq x \leq b$.
4. $Expo(\lambda)$:
   $f_X(x) = \lambda \exp\{-\lambda x\}1\{x \geq 0\}; F_X(x) = 1 - \exp\{-\lambda x\}$ for $x \leq 0$.
5. Target: $f_X(x) = 2x1\{0 \leq x \leq 1\}; F_X(x) = x^2$ for $0 \leq x \leq 1$.