CS70: Lecture 29

Continuous Probability (continued)

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Continuous Probability (continued)

1. Review: CDF, PDF
2. Examples
3. Properties
4. Expectation of continuous random variables

Review: Continuous Probability

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A Picture

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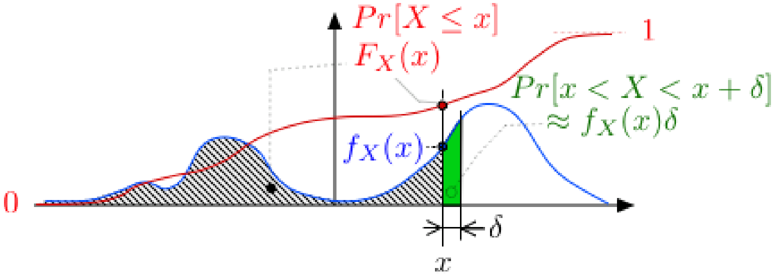

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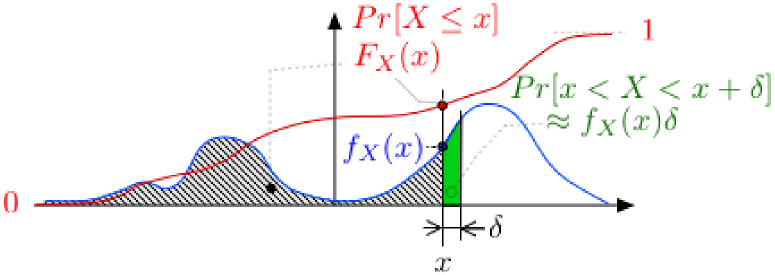

The pdf $f_X(x)$ is a nonnegative function that integrates to 1.

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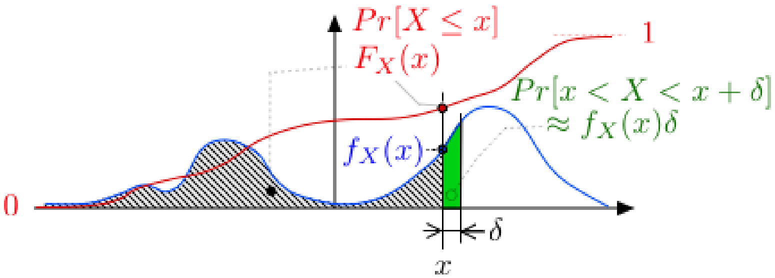

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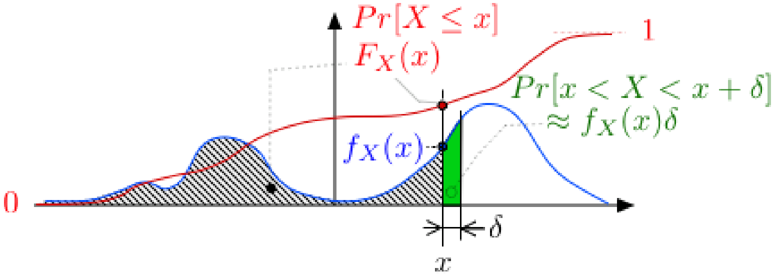

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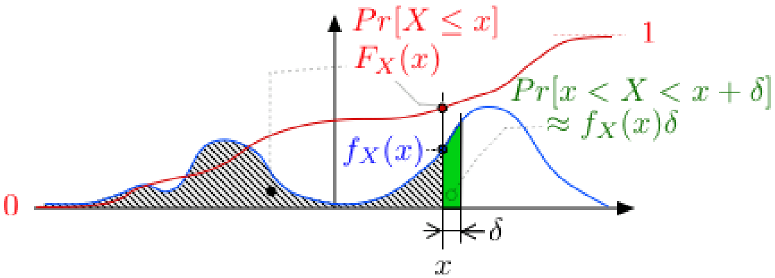

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$$Pr[X \leq x] = F_x(x) = \int_{-\infty}^{x} f_X(u)du$$

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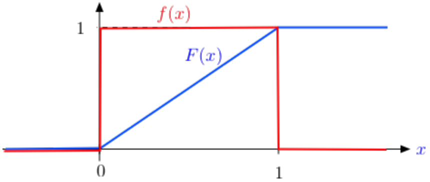

$$Pr[X \in (a, b]] = Pr[X \leq b] - Pr[X \leq a]$$

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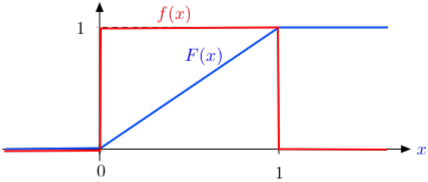

$$Pr[X \in (a,b]] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a).$$

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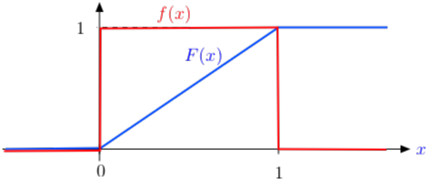

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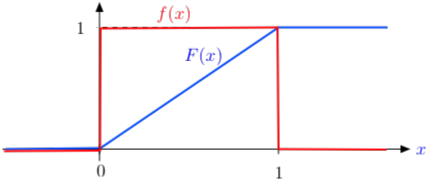

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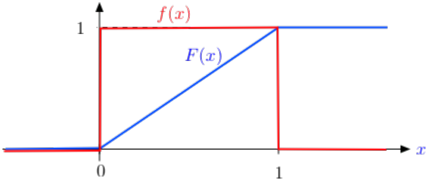

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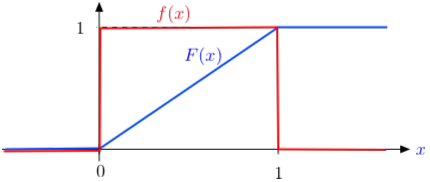

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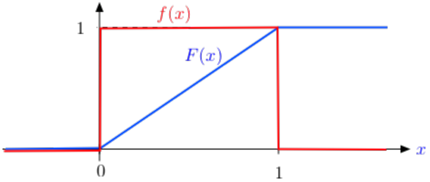

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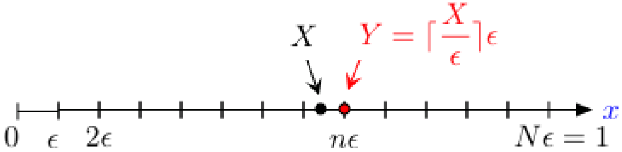

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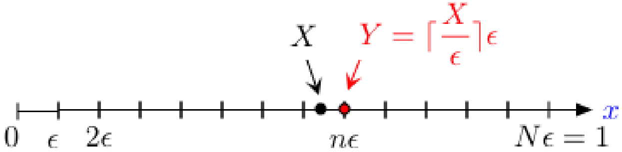

Discrete Approximation:

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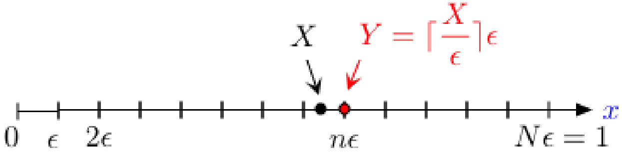

Discrete Approximation: Fix $N \gg 1$

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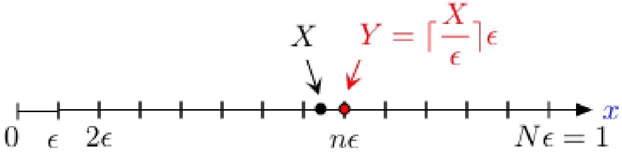

Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

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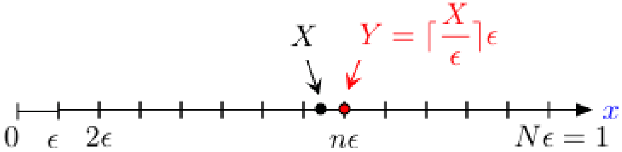

Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

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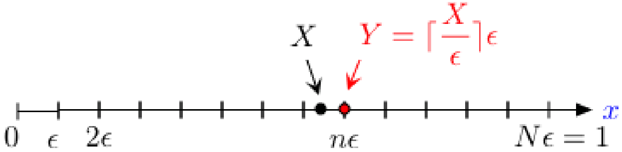

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Then $|X - Y| \leq \varepsilon$

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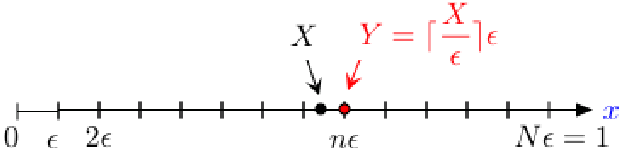

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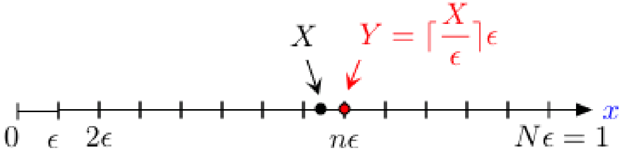

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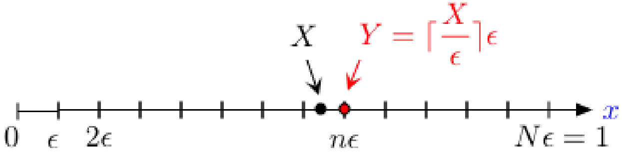

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Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

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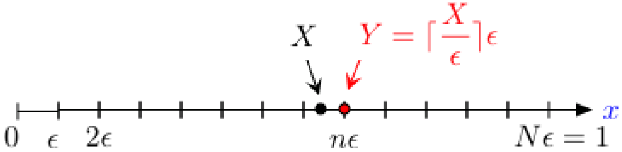

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Thus, X is 'almost discrete.'

Nonuniform Choice at Random in $[0, 1]$.

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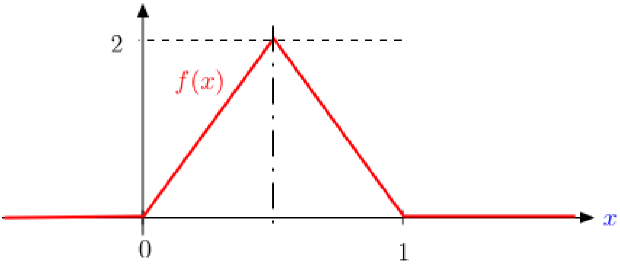

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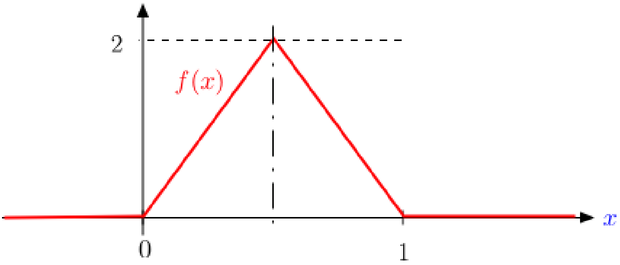

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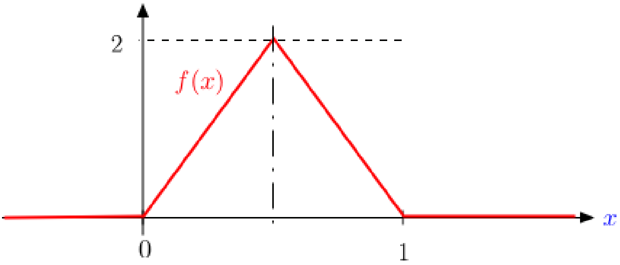

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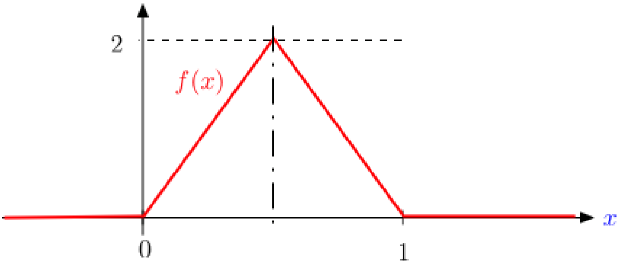

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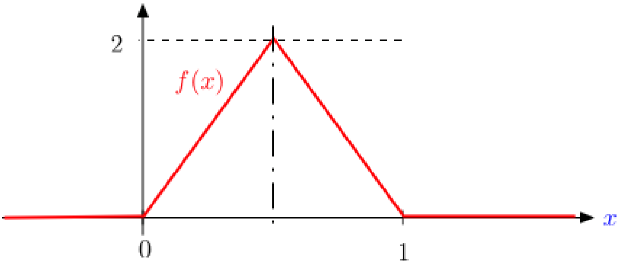

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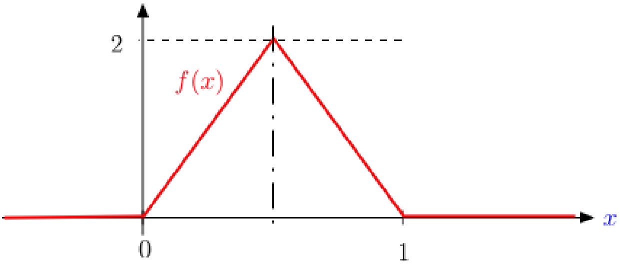

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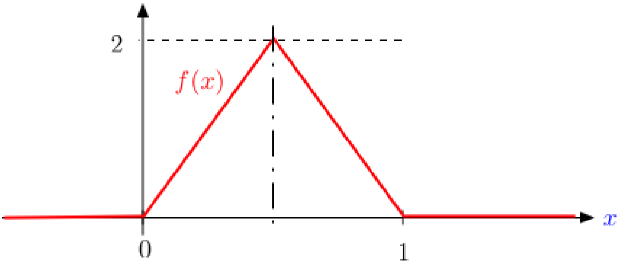

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For instance, $Pr[X \in [0, 1/3]] = \int_0^{1/3} 4xdx = 2\left[x^2\right]_0^{1/3} = \frac{2}{9}$.

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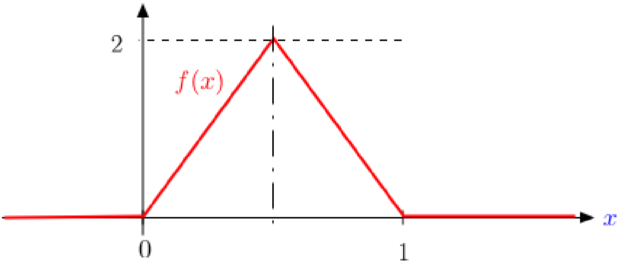

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Thus, $Pr[X \in [0, 1/3]] = Pr[X \in [2/3, 1]] = \frac{2}{9}$

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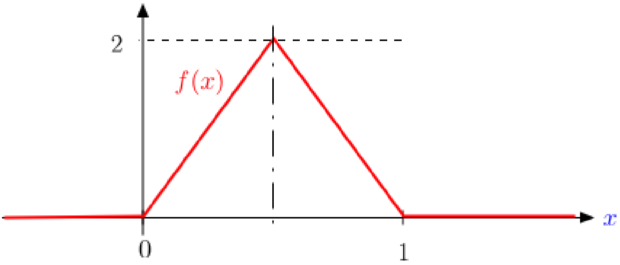

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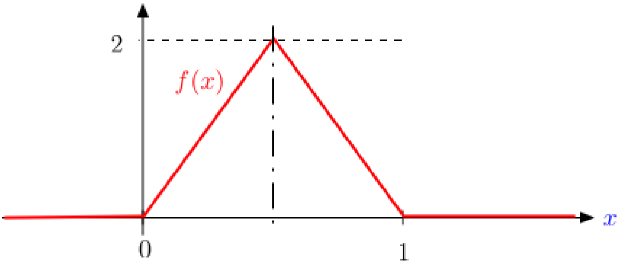
Thus, $Pr[X \in [0,1/3]] = Pr[X \in [2/3,1]] = \frac{2}{9}$ and $Pr[X \in [1/3,2/3]] =$

Nonuniform Choice at Random in $[0,1]$.



This figure shows yet a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx = 1$.

It defines another way of choosing X at random in $[0,1]$.

Note that X is more likely to be closer to $1/2$ than to 0 or 1.

For instance, $Pr[X \in [0,1/3]] = \int_0^{1/3} 4x dx = 2\left[x^2\right]_0^{1/3} = \frac{2}{9}$.

Thus, $Pr[X \in [0,1/3]] = Pr[X \in [2/3,1]] = \frac{2}{9}$ and $Pr[X \in [1/3,2/3]] = \frac{5}{9}$.

General Random Choice in $\mathfrak{R}$

General Random Choice in $\Re$

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To indicate that F and f correspond to the RV X, we will write them $F_X(x)$ and $f_X(x)$.

$Pr[X \in (x, x+\varepsilon)]$

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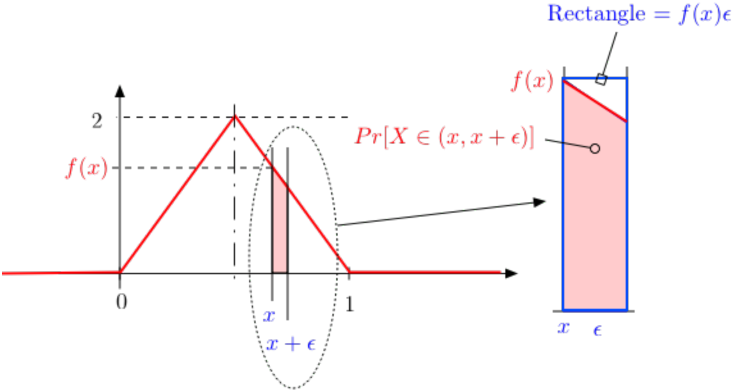

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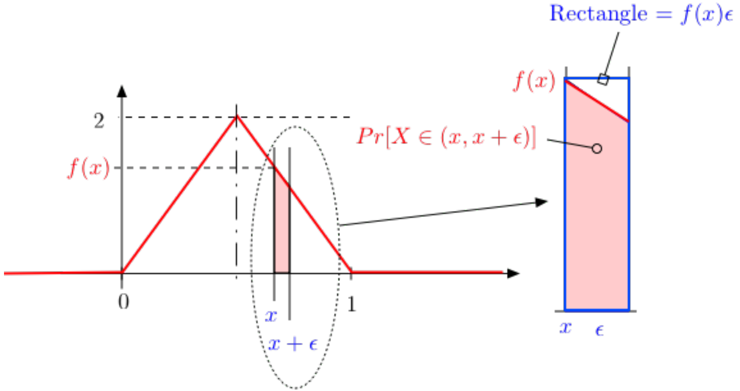

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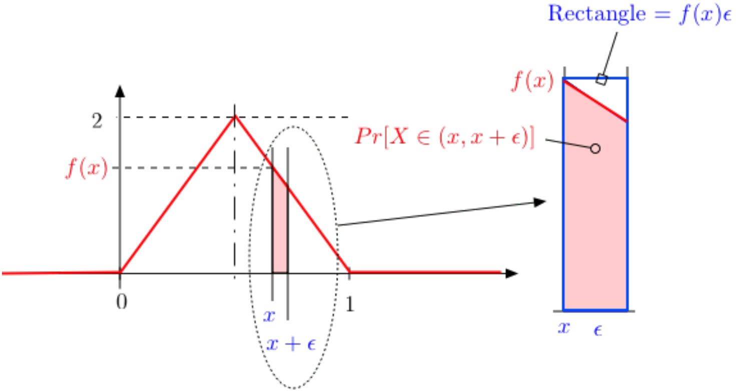

Thus, the pdf is the ‘local probability by unit length.’

It is the ‘probability density.’

Discrete Approximation

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Example: hitting random location on gas tank.

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Random location on circle.

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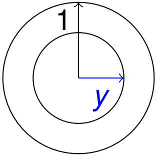

Random Variable: Y distance from center.

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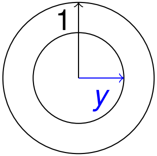

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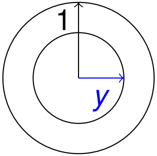

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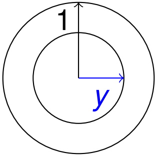

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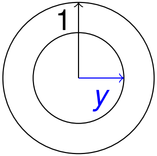

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Hence,

$$F_Y(y) = Pr[Y \leq y] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Calculation of event with dartboard..

Probability between .5 and .6 of center?

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PDF.

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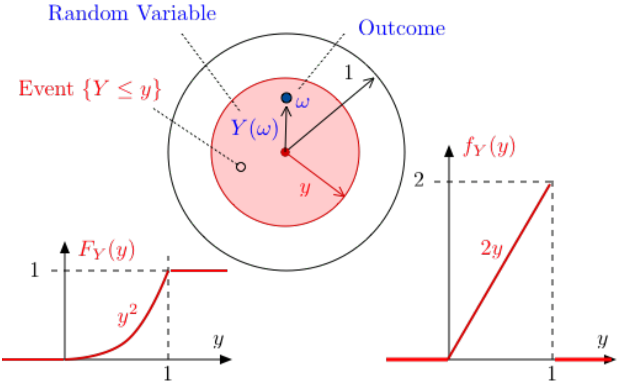
The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.
Use whichever is convenient.

Target

Target



Random Variable
Outcome
Event $\{Y \leq y\}$
1
ω
$Y(\omega)$
y
$f_Y(y)$
2
$F_Y(y)$
1
$2y$
y^2
y
1
y
1

$U[a, b]$

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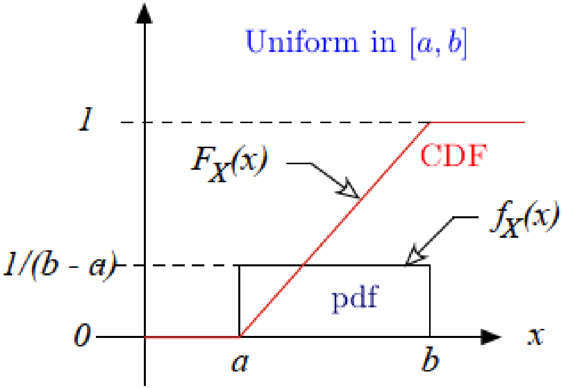

Expo(λ)

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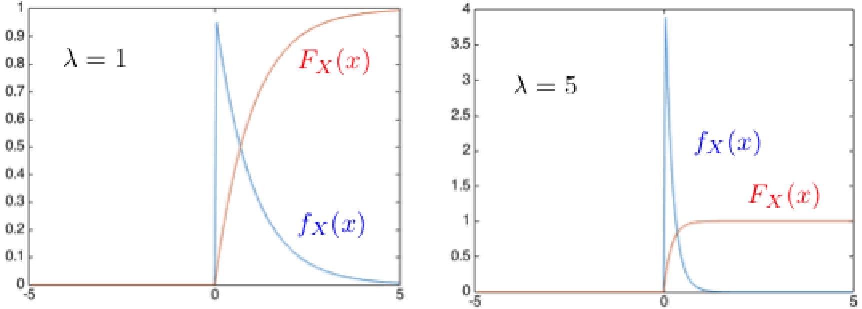

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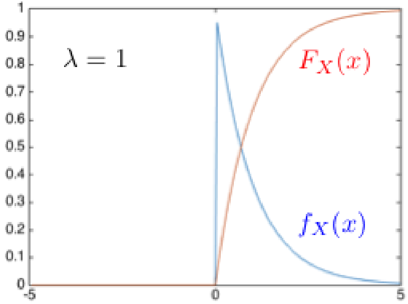

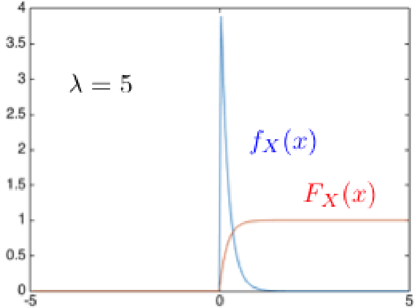

Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

Some Properties

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1. ***Expo* is memoryless.**

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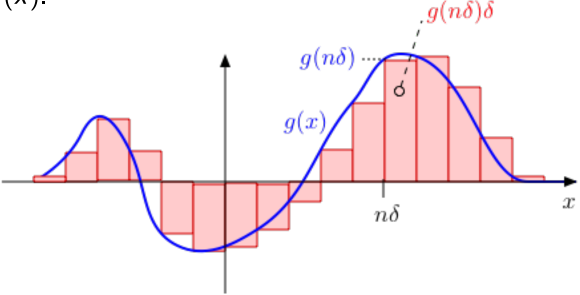

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