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Principle of Induction.

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Climb an infinite ladder?

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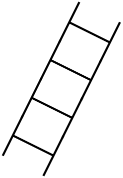

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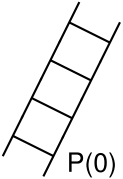
$P(0)$



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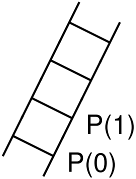

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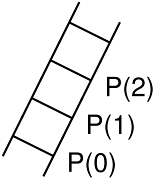

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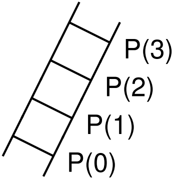

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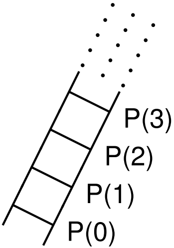

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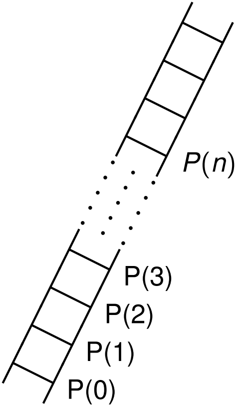

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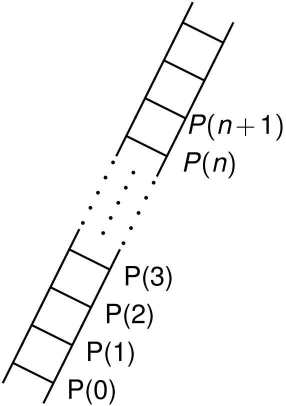

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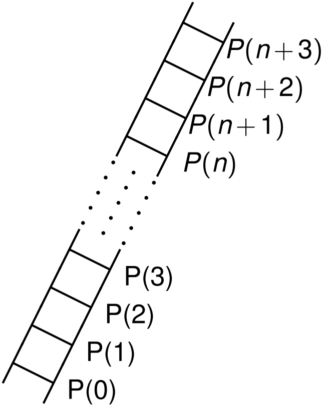

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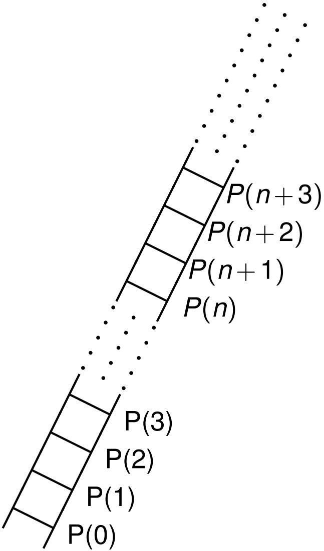

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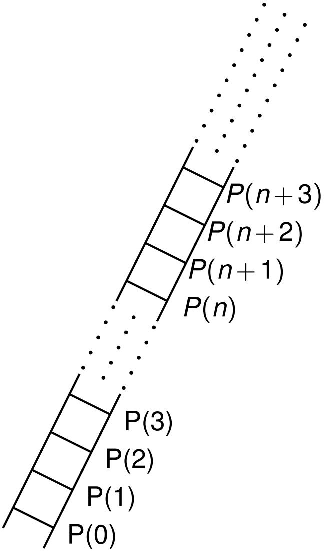

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Your favorite example of forever..

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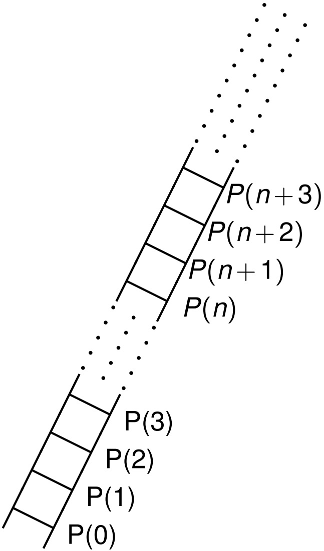

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Your favorite example of forever..or the natural numbers...

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

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$(k+1)^3 - (k+1)$

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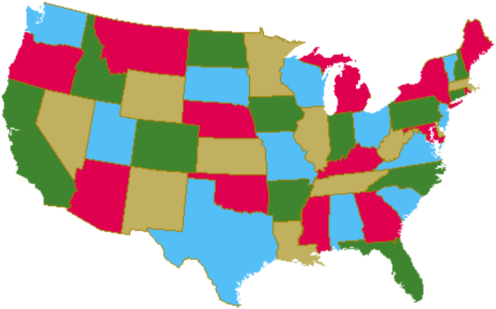
Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



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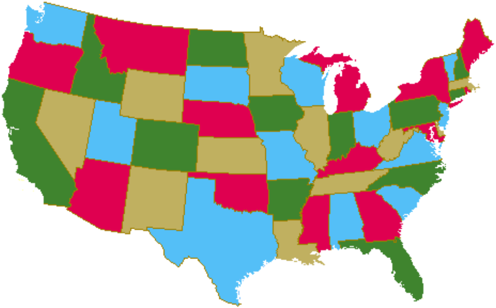

Check Out: "Four corners".

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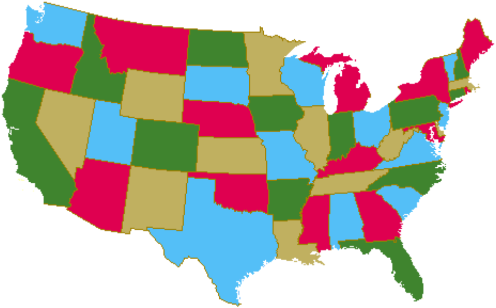

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States connected at a point, can have same color.

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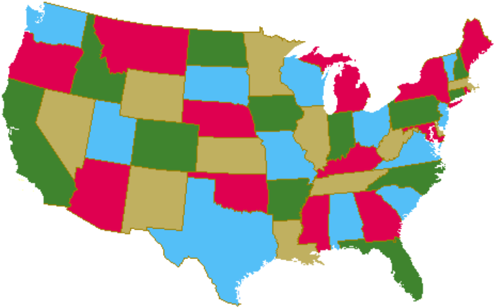

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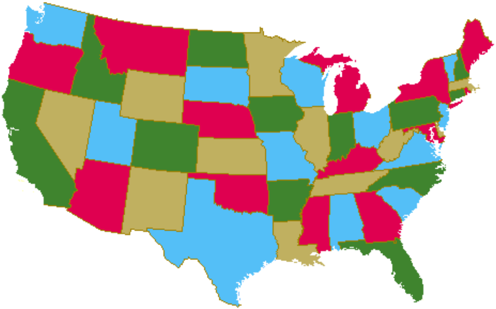

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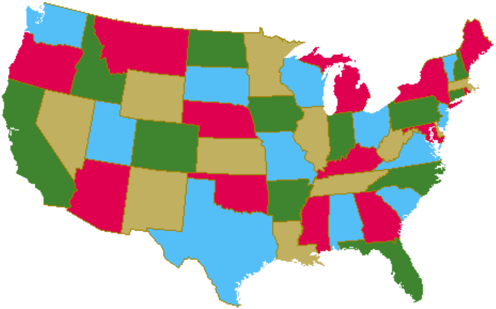

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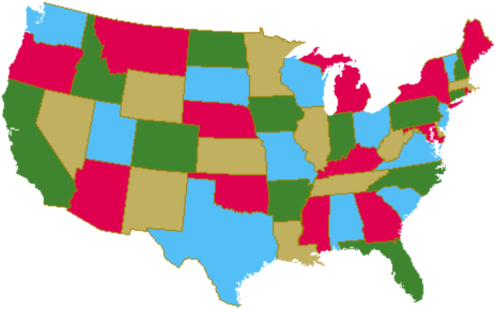

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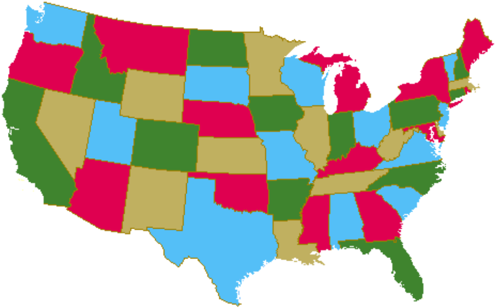

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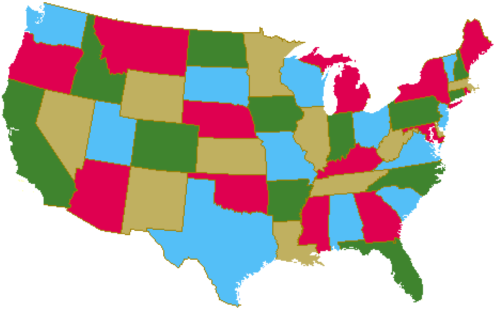

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Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

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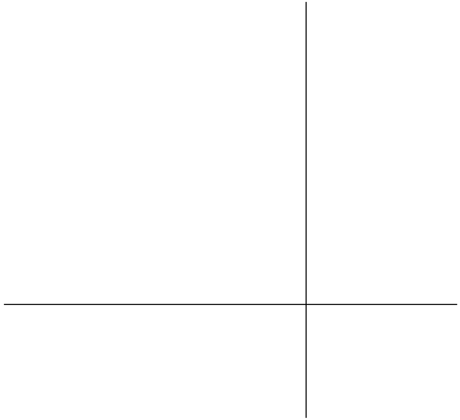

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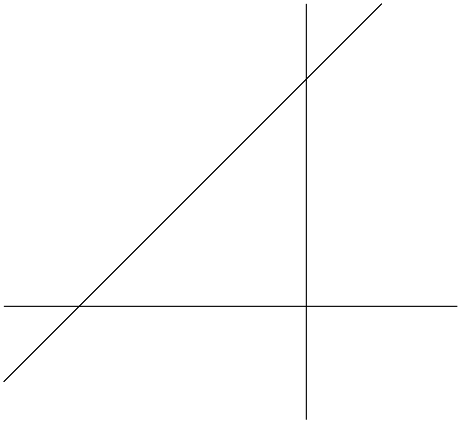

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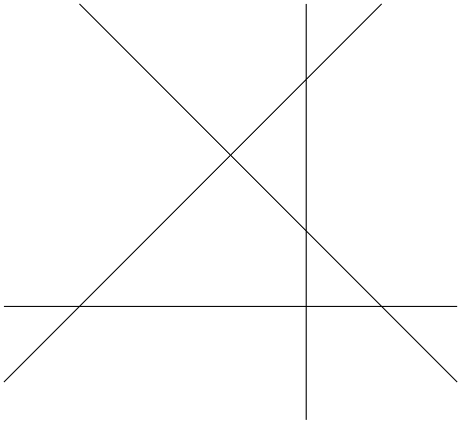

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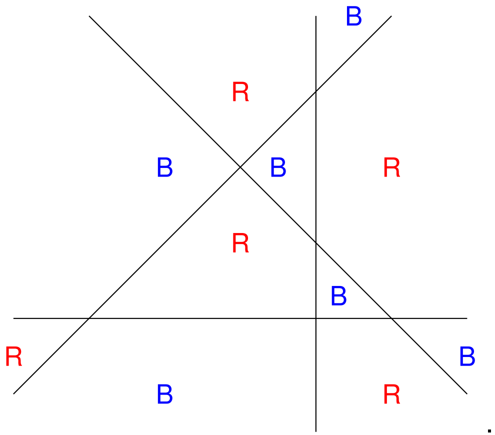

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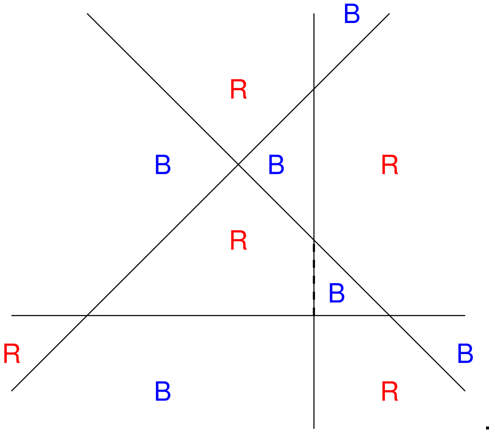

Fact: Swapping red and blue gives another valid colors.

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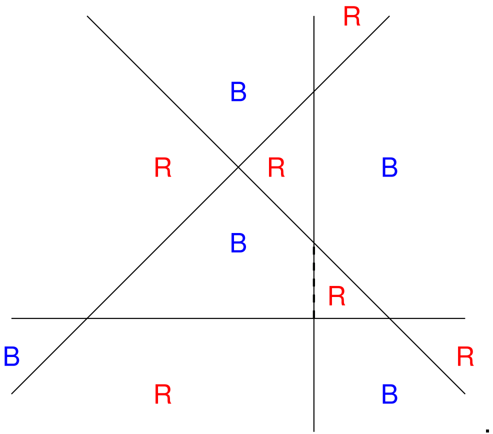

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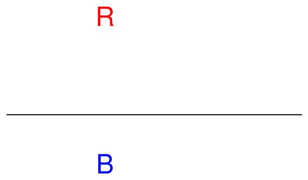
Two color theorem: proof illustration.

Base Case.

Two color theorem: proof illustration.



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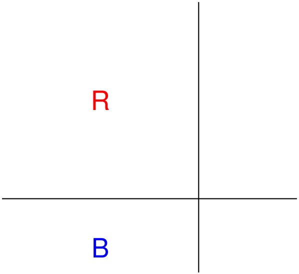
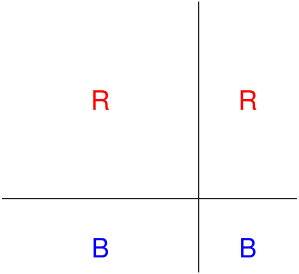


1. Add line.

Two color theorem: proof illustration.



1. Add line.
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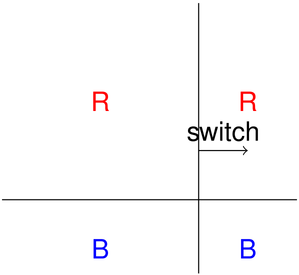
Two color theorem: proof illustration.



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(Fixes conflicts along line, and makes no new ones.)

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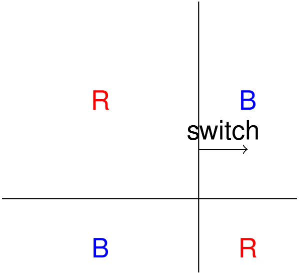

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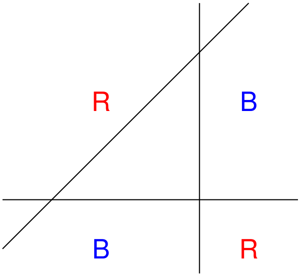

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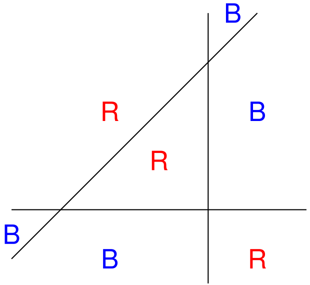

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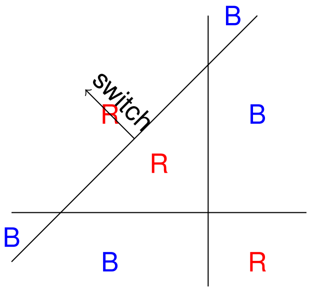

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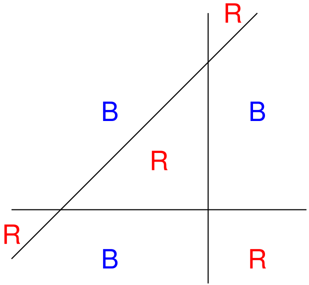

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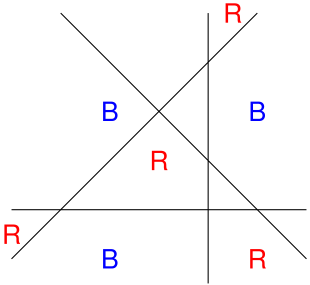

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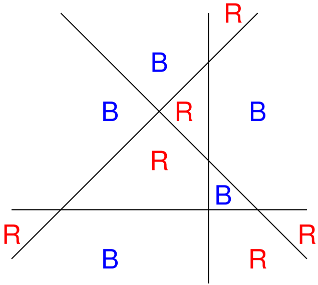

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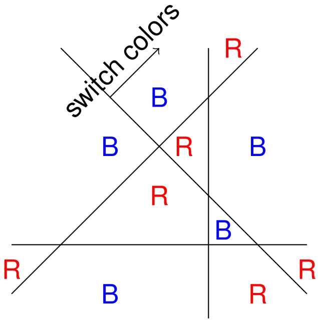

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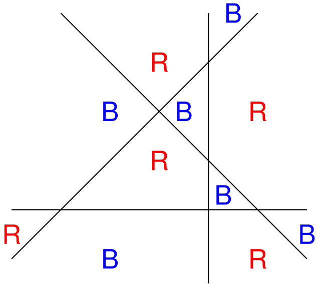

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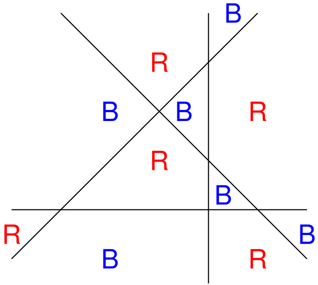

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Algorithm gives $P(k) \implies P(k+1)$.

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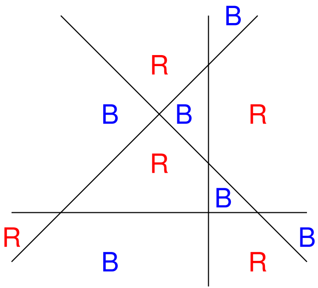

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Theorem: The sum of the first n odd numbers is a perfect square.

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kth odd number is $2(k-1)+1$.

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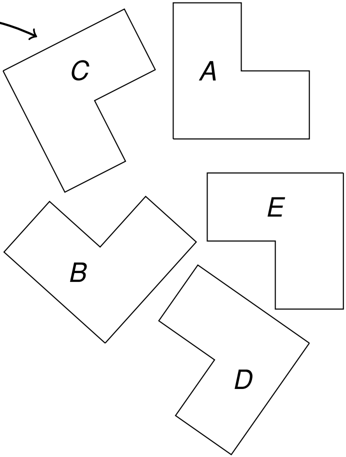
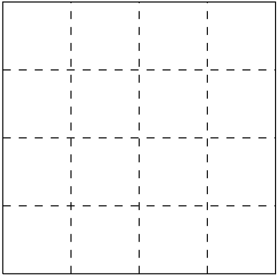
$\square$

Tiling Cory Hall Courtyard.

Use these *L*-tiles.



To Tile this 4×4 courtyard.



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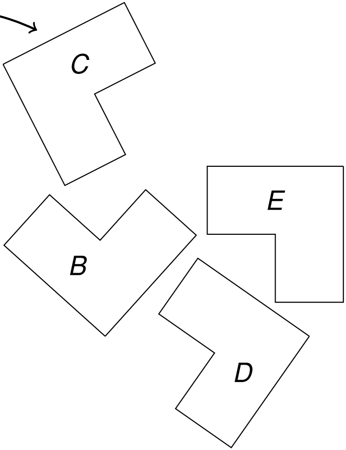

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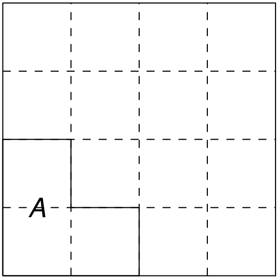

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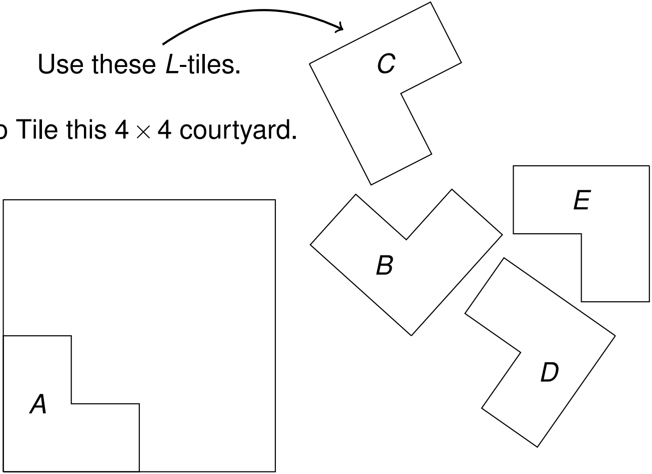

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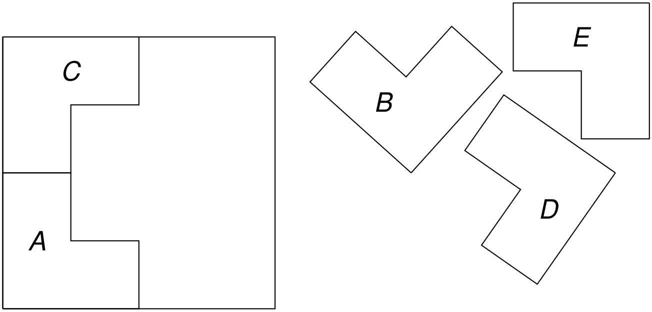

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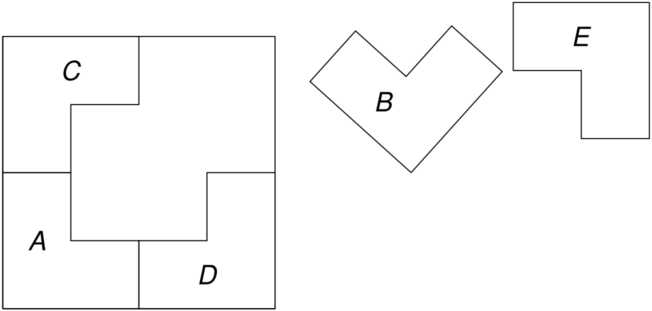

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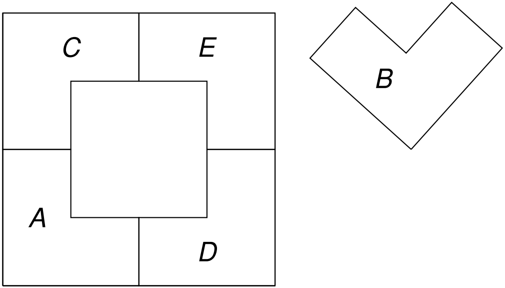

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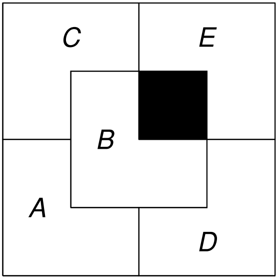

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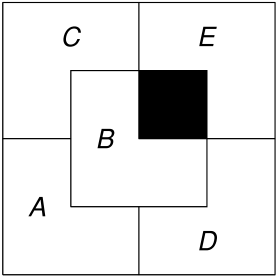

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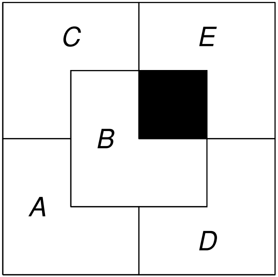
To Tile this 4×4 courtyard.



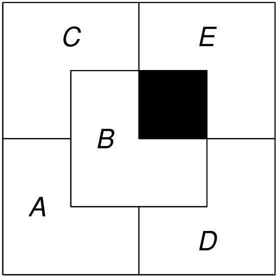
Alright!

Tiled 4×4 square with 2×2 L-tiles.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

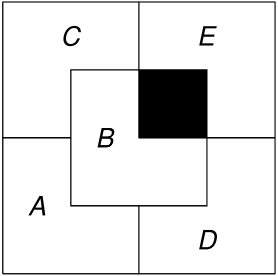
Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



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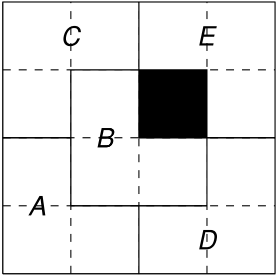
Can we tile any $2^n \times 2^n$ with L-tiles (with a hole)

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

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$$2^{2(k+1)} = 2^{2k} * 2^2$$

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$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \end{aligned}$$

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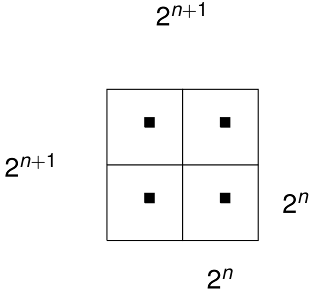

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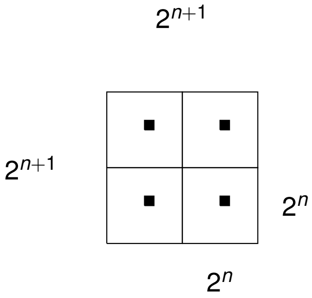

What to do now???

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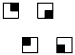

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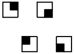

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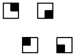

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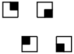

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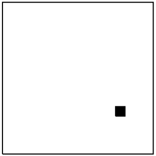

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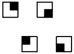
Flipping the orientation can leave hole anywhere. 

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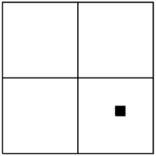

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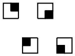

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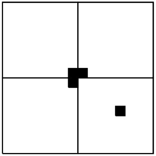

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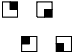

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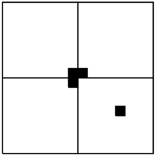

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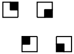
Flipping the orientation can leave hole anywhere. 

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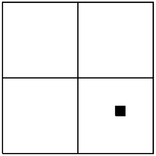

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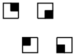

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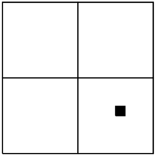

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E.g. Reduced form is “smallest” representation of a rational number a/b.

Well ordering principle.

Thm: All natural numbers are interesting.

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Thus, there is no smallest uninteresting natural number.

Thus: All natural numbers are interesting.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

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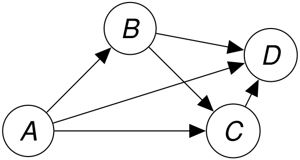

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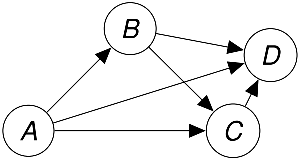

Theorem: Any tournament that has a cycle has a cycle of length 3.

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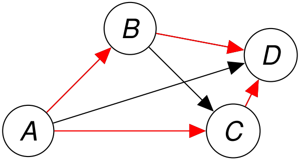

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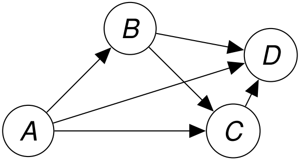

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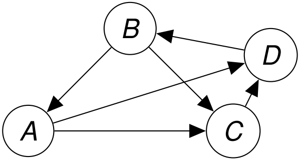

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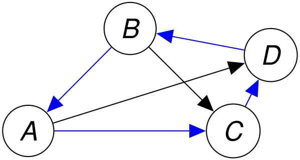

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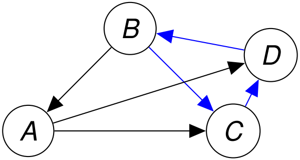

Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournament has a cycle of length 3 if at all.

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Assume the the **smallest cycle** is of length k.

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Case 1: Of length 3.

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Case 1: Of length 3. Done.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

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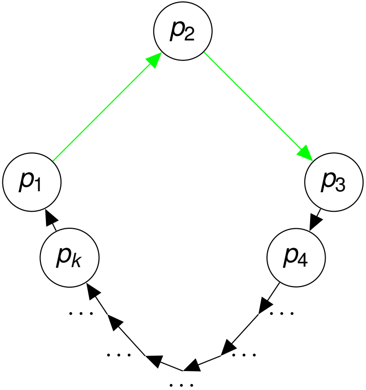
Case 2: Of length larger than 3.



"$p_3 \rightarrow p_1$" $\implies$ 3 cycle

Contradiction.

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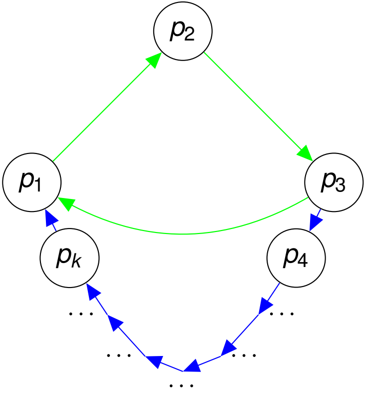

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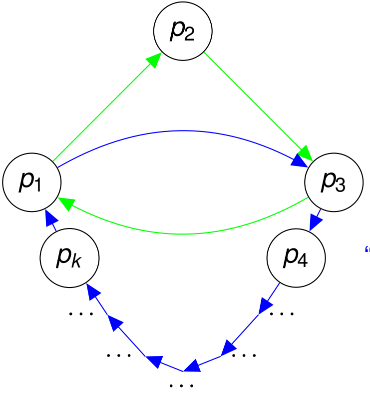

"$p_3 \to p_1$" $\Longrightarrow$ 3 cycle

Contradiction.

"$p_1 \to p_3$" $\Longrightarrow$ $k-1$ length cycle!

Contradiction!

Tournaments have long paths.

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$p_1, \ldots, p_n$, $(\forall i, 0 \leq i < n)$ $p_i \to p_{i+1}$.

Base: True for two vertices.
(Also for one, but two is more useful as base case!)

Tournament on $n+1$ people,
Remove arbitrary person $\to$ yield tournament on $n-1$ people.
(Result specified for each remaining pair from original tournament.)

By induction hypothesis: There is a sequence $p_1, \ldots, p_n$
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Horses of the same color...

Theorem: All horses have the same color.

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As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

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Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

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Island with 100 possibly blue-eyed and green-eyed inhabitants.

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Wait! Visitor added no information.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

Common Knowledge.

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On day 1, everyone knows everyone sees more than zero.

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On day 99, no one sees 98

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

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On day 2, everyone knows everyone sees more than one.

...

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100,

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

…

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!
  No one knows other people see that he has no clothes.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!
  No one knows other people see that he has no clothes.
  Until kid points it out.

Summary: principle of induction.

Today: More induction.

Summary: principle of induction.

Today: More induction.

$(P(0)$

Summary: principle of induction.

Today: More induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1))))$$

Summary: principle of induction.

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$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Summary: principle of induction.

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$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0

Summary: principle of induction.

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$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

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Base Case: Prove $P(n_0)$.

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Base Case: Prove $P(n_0)$.
Ind. Step: Prove.

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$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

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Statement is proven!

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Strong Induction:

Summary: principle of induction.

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Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1))))$

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

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Base Case: Prove $P(n_0)$.
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Statement is proven!

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Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Summary: principle of induction.

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Strengthen theorem statement.

Summary: principle of induction.

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Strengthen theorem statement.
Sum of first n odds is n^2.

Summary: principle of induction.

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Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.

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Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction.

Summary: principle of induction.

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Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction. E.g., used in product of primes proof.

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Induction $\equiv$ Recursion.

Summary: principle of induction.

$(P(0)$

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Variations:

$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

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$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

$$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1))))$$

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$$\begin{aligned}&(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1)))) \\ &\qquad \implies (\forall n \in N)((n \geq 1) \implies P(n))\end{aligned}$$

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Base Case: Prove $P(n_0)$.
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Statement is proven!