CS70: Lecture 30

Continuous Probability (contd.)

1. Review: CDF, PDF
2. Review: Expectation
3. Review: Independence
4. Meeting at a Restaurant
5. Breaking a Stick
6. Maximum of Exponentials
7. Geometric and Exponential

Review: CDF and PDF.

Key idea: For a continuous RV, $Pr[X = x] = 0$ for all $x \in \Re$.

Examples: Uniform in $[0,1]$; throw a dart in a target.

Thus, one cannot define $Pr[\text{outcome}]$, then $Pr[\text{event}]$.

Instead, one starts by defining $Pr[\text{event}]$.

Thus, one defines $Pr[X \in (-\infty, x]] = Pr[X \le x] =: F_X(x), x \in \Re$.

Then, one defines $f_X(x) := \frac{d}{dx} F_X(x)$.

Hence, $f_X(x)\varepsilon = Pr[X \in (x, x+\varepsilon)]$.

$F_X(\cdot)$ is the cumulative distribution function (CDF) of X.

$f_X(\cdot)$ is the probability density function (PDF) of X.

Expectation

Definitions: (a) The **expectation** of a random variable X with pdf $f(x)$ is defined as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

(b) The expectation of a function of a random variable is defined as

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx.$$

(c) The expectation of a function of multiple random variables is defined as

$$E[h(\mathbf{X})] = \int \cdots \int h(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) dx_1 \cdots dx_n.$$

Justifications: Think of the discrete approximations of the continuous RVs.

Multiple Continuous Random Variables

One defines a pair (X, Y) of continuous RVs by specifying $f_{X,Y}(x,y)$ for $x, y \in \Re$ where

$$f_{X,Y}(x,y)dxdy = Pr[X \in (x, x+dx), Y \in (y+dy)].$$

The function $f_{X,Y}(x,y)$ is called the joint pdf of X and Y.

Example: Choose a point (X,Y) uniformly in the set $A \subset \Re^2$. Then

$$f_{X,Y}(x,y) = \frac{1}{|A|} 1\{(x,y) \in A\}$$

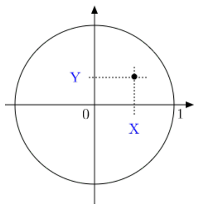
where $|A|$ is the area of A.

Interpretation. Think of (X,Y) as being discrete on a grid with mesh size ε and $Pr[X = m\varepsilon, Y = n\varepsilon] = f_{X,Y}(m\varepsilon, n\varepsilon)\varepsilon^2$.

Extension: $\mathbf{X} = (X_1, \ldots, X_n)$ with $f_{\mathbf{X}}(\mathbf{x})$.

Example of Continuous (X, Y)

Pick a point (X, Y) uniformly in the unit circle.



Thus, $f_{X,Y}(x,y) = \frac{1}{\pi} 1\{x^2 + y^2 \le 1\}$.

Consequently,

$$Pr[X > 0, Y > 0] = \frac{1}{4}$$
$$Pr[X < 0, Y > 0] = \frac{1}{4}$$
$$Pr[X^2 + Y^2 \le r^2] = r^2$$
$$Pr[X > Y] = \frac{1}{2}.$$

Independent Continuous Random Variables

Definition: The continuous RVs X and Y are independent if

$$Pr[X \in A, Y \in B] = Pr[X \in A] Pr[Y \in B], \forall A, B.$$

Theorem: The continuous RVs X and Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

Proof: As in the discrete case.

Definition: The continuous RVs $X_1, \ldots, X_n$ are mutually independent if

$$Pr[X_1 \in A_1, \ldots, X_n \in A_n] = Pr[X_1 \in A_1] \cdots Pr[X_n \in A_n], \forall A_1, \ldots, A_n.$$

Theorem: The continuous RVs $X_1, \ldots, X_n$ are mutually independent if and only if

$$f_{\mathbf{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n).$$

Proof: As in the discrete case.

Examples of Independent Continuous RVs

1. Minimum of Independent *Expo*. Let $X = Expo(\lambda)$ and $Y = Expo(\mu)$ be independent RVs.

Recall that $Pr[X > u] = e^{-\lambda u}$. Then

$$\begin{aligned} Pr[\min\{X,Y\} > u] &= Pr[X > u, Y > u] = Pr[X > u]Pr[Y > u] \\ &= e^{-\lambda u} \times e^{-\mu u} = e^{-(\lambda+\mu)u}. \end{aligned}$$

This shows that $\min\{X,Y\} = Expo(\lambda+\mu)$.

Thus, the minimum of two independent exponentially distributed RVs is exponentially distributed.

2. Minimum of Independent $U[0,1]$**.** Let $X, Y = [0,1]$ be independent RVs. Let also $Z = \min\{X,Y\}$. What is f_Z?

One has

$$Pr[Z > u] = Pr[X > u]Pr[Y > u] = (1-u)^2.$$

Thus $F_Z(u) = Pr[Z \le u] = 1 - (1-u)^2$.

Hence, $f_Z(u) = \frac{d}{du}F_Z(u) = 2(1-u), u \in [0,1]$. In particular, $E[Z] = \int_0^1 u f_Z(u)du = \int_0^1 2u(1-u)du = 2\frac{1}{2} - 2\frac{1}{3} = \frac{1}{3}$.

Expectation of Product of Independent RVs

Theorem If X, Y, X are mutually independent, then

$$E[XYZ] = E[X]E[Y]E[Z].$$

Proof: Same as discrete case.

Example: Let X, Y, Z be mutually independent and $U[0,1]$. Then

$$\begin{aligned} E[(X+2Y+3Z)^2] &= E[X^2+4Y^2+9Z^2+4XY+6XZ+12YZ] \\ &= \frac{1}{3}+4\frac{1}{3}+9\frac{1}{3}+4\frac{1}{2}\frac{1}{2}+6\frac{1}{2}\frac{1}{2}+12\frac{1}{2}\frac{1}{2} \\ &= \frac{14}{3}+\frac{22}{4} \approx 10.17. \end{aligned}$$

Variance

Definition: The **variance** of a continuous random variable X is defined as

$$var[X] = E((X - E(X))^2) = E(X^2) - (E(X))^2.$$

Example 1: $X = U[0,1]$. Then

$$var[X] = E[X^2] - E[X]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Example 2: $X = Expo(\lambda)$. Then $E[X] = \lambda^{-1}$ and $E[X^2] = 2/(\lambda^2)$.

Hence, $var[X] = 1/(\lambda^2)$.

Example 3: Let X, Y, Z be independent. Then

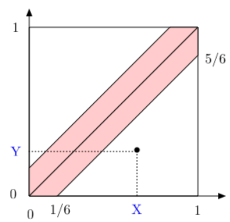
$$var[X+Y+Z] = var[X] + var[Y] + var[Z],$$

as in the discrete case.

Meeting at a Restaurant

Two friends go to a restaurant independently uniformly at random between noon and 1pm.

They agree they will wait for 10 minutes. What is the probability they meet?



Here, (X, Y) are the times when the friends reach the restaurant.

The shaded area are the pairs where $|X - Y| < 1/6$, i.e., such that they meet.

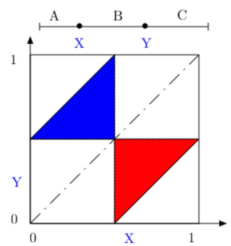
The complement is the sum of two rectangles. When you put them together, they form a square with sides $5/6$.

Thus, $Pr[\text{meet}] = 1 - (\frac{5}{6})^2 = \frac{11}{36}$.

Breaking a Stick

You break a stick at two points chosen independently uniformly at random.

What is the probability you can make a triangle with the three pieces?



Let X, Y be the two break points along the $[0,1]$ stick.

You can make a triangle if $A < B+C, B < A+C$, and $C < A+B$.

If $X < Y$, this means $X < 0.5, Y < X + 0.5, Y > 0.5$. This is the blue triangle.

If $X > Y$, we get the red triangle, by symmetry.

Thus, $Pr[\text{make triangle}] = 1/4$.

Maximum of Two Exponentials

Let $X = Expo(\lambda)$ and $Y = Expo(\mu)$ be independent. Define $Z = \max\{X,Y\}$.

Calculate $E[Z]$.

We compute f_Z, then integrate.

One has

$$\begin{aligned} Pr[Z < z] &= Pr[X < z, Y < z] = Pr[X < z]Pr[Y < z] \\ &= (1-e^{-\lambda z})(1-e^{-\mu z}) = 1 - e^{-\lambda z} - e^{-\mu z} + e^{-(\lambda+\mu)z} \end{aligned}$$

Thus,

$$f_Z(z) = \lambda e^{-\lambda z} + \mu e^{-\mu z} - (\lambda+\mu)e^{-(\lambda+\mu)z}, \forall z > 0.$$

Hence,

$$E[Z] = \int_0^\infty z f_Z(z)dz = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}.$$

Maximum of n i.i.d. Exponentials

Let $X_1,\dots,X_n$ be i.i.d. $Expo(1)$. Define $Z = \max\{X_1, X_2, \dots, X_n\}$. Calculate $E[Z]$.

We use a recursion. The key idea is as follows:

$$Z = \min\{X_1,\dots,X_n\} + V$$

where V is the maximum of $n-1$ i.i.d. $Expo(1)$. This follows from the memoryless property of the exponential.

Let then $A_n = E[Z]$. We see that

$$\begin{aligned} A_n &= E[\min\{X_1,\dots,X_n\}] + A_{n-1} \\ &= \frac{1}{n} + A_{n-1} \end{aligned}$$

because the minimum of *Expo* is *Expo* with the sum of the rates.

Hence,

$$E[Z] = A_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H(n).$$

Geometric and Exponential

The geometric and exponential distributions are similar. They are both memoryless.

Consider flipping a coin every $1/N$ second with $Pr[H] = p/N$, where $N \gg 1$.

Let X be the time until the first H.

Fact: $X \approx Expo(p)$.

Analysis: Note that

$$\begin{aligned} Pr[X > t] &\approx Pr[\text{first } Nt \text{ flips are tails}] \\ &= (1 - \frac{p}{N})^{Nt} \approx \exp\{-pt\}. \end{aligned}$$

Indeed, $(1 - \frac{a}{N})^N \approx \exp\{-a\}$.

Summary

Continuous Probability

- Continuous RVs are essentially the same as discrete RVs
- Think that $X \approx x$ with probability $f_X(x)\varepsilon$
- Sums become integrals, ...
- The exponential distribution is magical: memoryless.