# CS70: Lecture 30

Continuous Probability (contd.)

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1. Review: CDF, PDF
2. Review: Expectation
3. Review: Independence
4. Meeting at a Restaurant
5. Breaking a Stick
6. Maximum of Exponentials
7. Geometric and Exponential

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**Justifications:** Think of the discrete approximations of the continuous RVs.

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**Extension:** $\mathbf{X} = (X_1, \ldots, X_n)$ with $f_{\mathbf{X}}(\mathbf{x})$.

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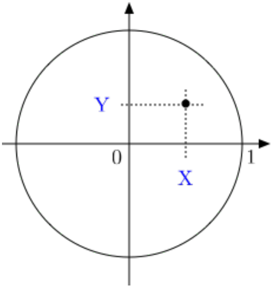

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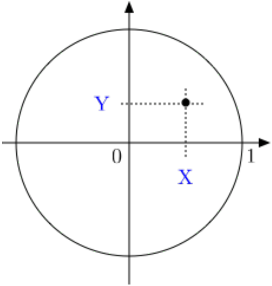


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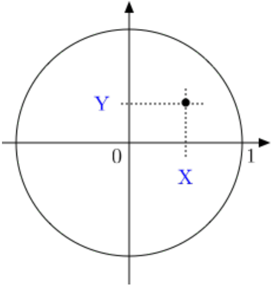


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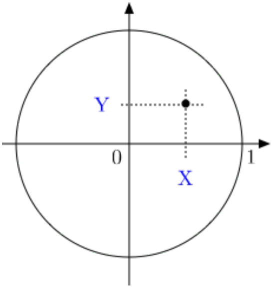


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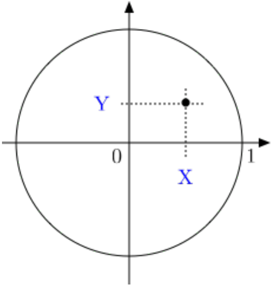


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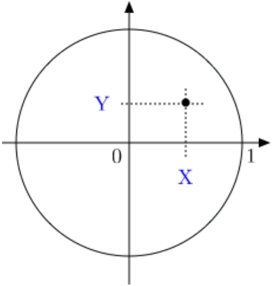


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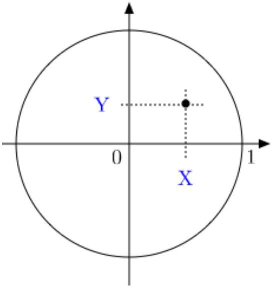


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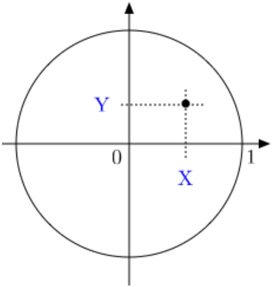


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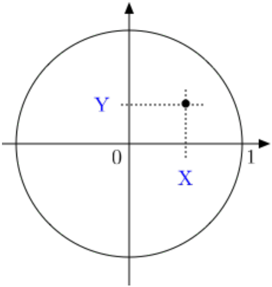


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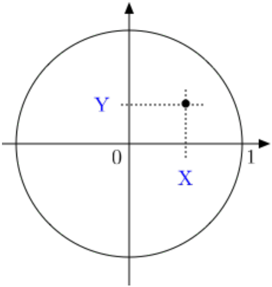


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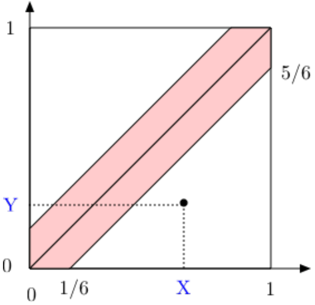

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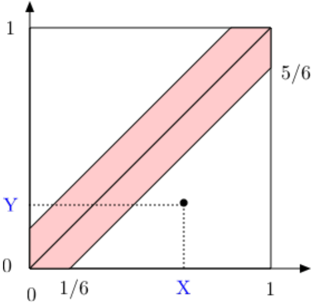


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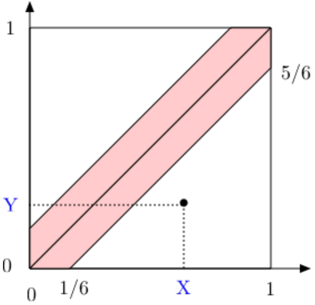


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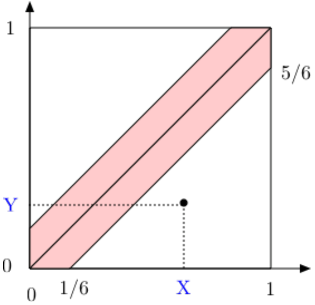


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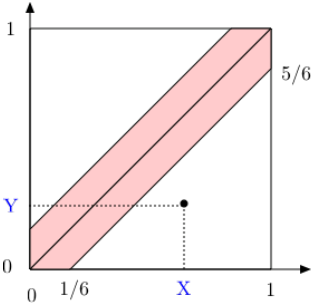


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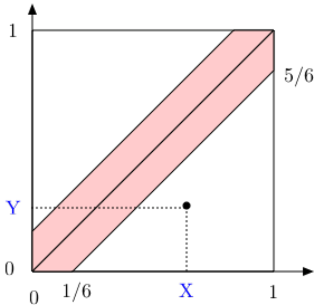


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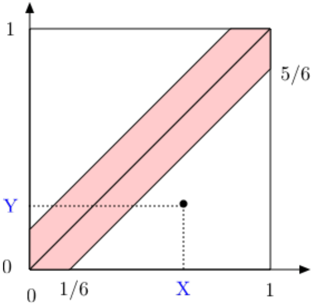


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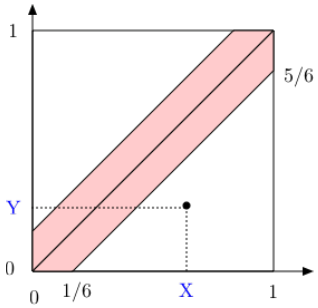


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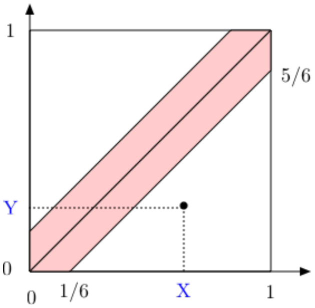


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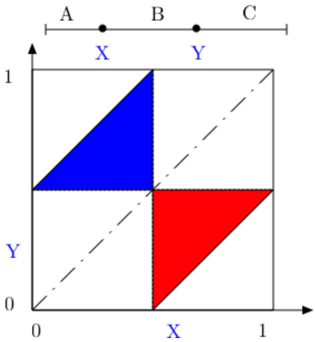

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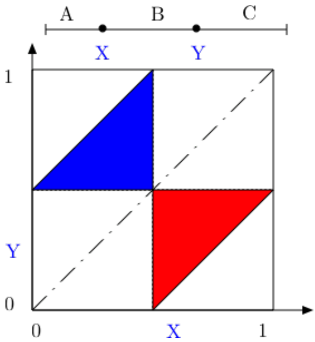


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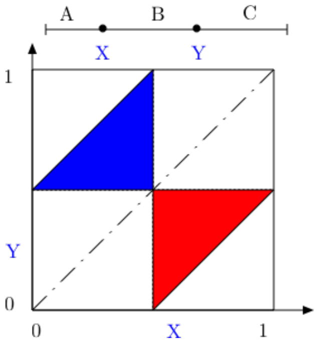


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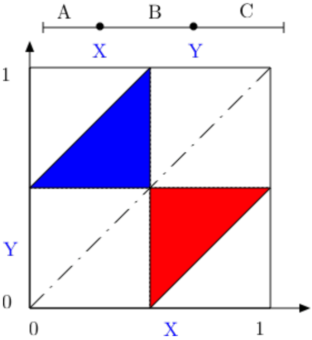


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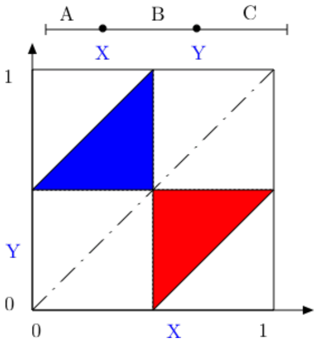


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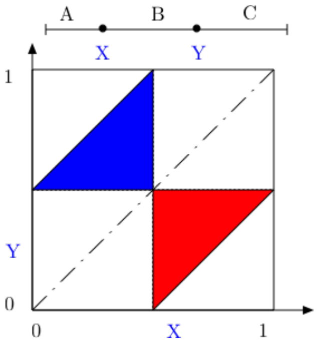


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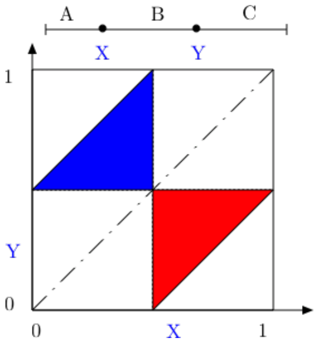


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We use a recursion. The key idea is as follows:

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We use a recursion. The key idea is as follows:

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Hence,

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Indeed, $(1 - \frac{a}{N})^N \approx \exp\{-a\}$.

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