CS70: Lecture 33.

$\boxed{\text{WLLN, Confidence Intervals (CI): Chebyshev vs. CLT}}$

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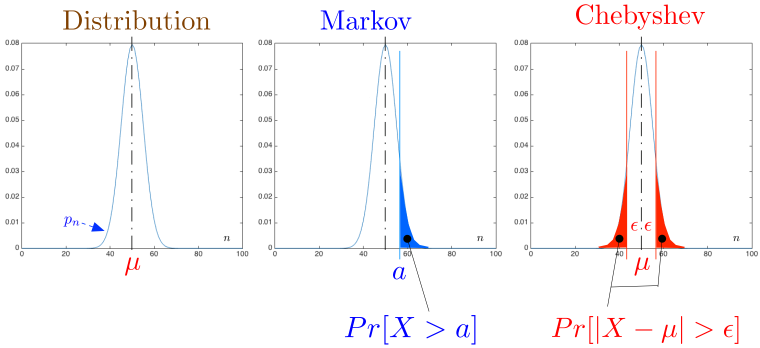
WLLN, Confidence Intervals (CI): Chebyshev vs. CLT

1. Review: Inequalities: Markov, Chebyshev
2. Law of Large Numbers
3. Review: CLT
4. Confidence Intervals: Chebyshev vs. CLT

Inequalities: An Overview



Markov Inequality

If X can only take non-negative values then

$$P(X \geq a) \leq \frac{E[X]}{a}$$

for all $a > 0$.

This inequality makes no assumptions on the existence of variance and so it can't be very strong for typical distributions. In fact, it is quite weak.

Chebyshev Inequality

If X is a random variable with finite mean and variance σ^2, then

$$P(|X - E[X]| \geq c) \leq \frac{\sigma^2}{c^2}$$

for all $c > 0$.
Also, letting $c = k\sigma$:

$$P(|X - E[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

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$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

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$$Var(X_i) = p(1-p) \leq (.5)(.5) = \frac{1}{4}$$

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We look at a general case next.

Weak Law of Large Numbers

We perform an experiment n times independently and

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The fact that $var(M_n) \to 0$ at rate $\frac{1}{n}$ is great but what does that tell us about $P(|M_n - E[X_i|])$? How quickly does it go to zero?
Just use Chebyshev: $P(|X - E[X]| \geq c) \leq \frac{\sigma^2}{c^2}$

$$P(|M_n - E[X_i]) \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

for any $\epsilon > 0$.
This is a form of the Weak Law of Large Numbers.

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$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

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$\square$

What does the Weak Law Really Mean?

WLLN: $\lim_{n\to\infty} P(|M_n - \mu| \geq \epsilon) = 0$.
Just using the defn of limit: For any $\epsilon, \delta > 0$, there exists a number $n(\epsilon, \delta)$ such that

$$P(|M_n - \mu| \geq \epsilon) \leq \delta \quad \text{for all } n \geq n(\epsilon, \delta)$$

- δ:Confidence level
- ϵ: "Error"
- $n(\epsilon, \delta)$: threshold function for a given level of confidence and accuracy

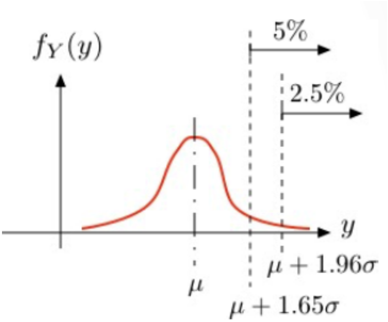
What this is saying is that if we compute M_n for large n then:
Almost Always, $|M_n - \mu| < \epsilon$.
We say that M_n **converges to** μ **in probability**.

Recap: Normal (Gaussian) Distribution.

For any μ and σ, a **normal** (aka **Gaussian**) random variable Y, which we write as $Y = \mathcal{N}(\mu, \sigma^2)$, has pdf

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2}.$$

Standard normal has $\mu = 0$ and $\sigma = 1$.



Note: $Pr[|Y - \mu| > 1.65\sigma] = 10\%$; $Pr[|Y - \mu| > 2\sigma] = 5\%$.

Recap: Central Limit Theorem

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Central Limit Theorem
Let $X_1, X_2, \ldots$ be i.i.d. with $E[X_1] = \mu$ and $var(X_1) = \sigma^2$. Define

$$S_n := \frac{A_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}.$$

Then,

$$S_n \to \mathcal{N}(0,1), \text{as } n \to \infty.$$

That is,

$$Pr[S_n \leq \alpha] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

$$E(S_n) = \frac{1}{\sigma/\sqrt{n}}(E(A_n) - \mu) = 0$$

$$Var(S_n) = \frac{1}{\sigma^2/n} Var(A_n) = 1.$$

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The CLT states that

$$\frac{A_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

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$$Pr[-2 \leq (\frac{A_n - \mu}{\sigma/\sqrt{n}}) \leq 2] \approx 95\%.$$

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CI for Mean: CLT vs. Chebyshev

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Thus, the CLT provides a smaller confidence interval.

Coins and CLT.

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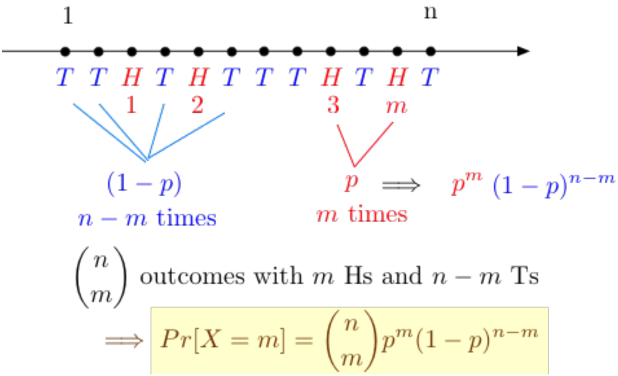

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Comparing Chebyshev and CLT: Polling

We ask n randomly sampled voters whether they support Bob.
$X_i = 1$ if the i^{th} voter says "yes" and $X_i = 0$ otherwise. The X_i are iid.
We want to be sure with prob ≥ 0.95 that $|M_{100} - p| \leq 0.1$. How many people should we ask?
Again, use the bound that $\mathrm{var}(X_i) \leq \frac{1}{4}$
By Chebyshev:

$$\frac{25}{n} \leq 0.05 \Rightarrow \boxed{n \geq 500}$$

By CLT:

$$2(1 - \phi(2 * 0.1 * \sqrt{n})) \leq 0.05$$

$$\phi(2 * 0.1 * \sqrt{n}) \geq 0.975$$

Since $\phi(1.96) = 0.975$:

$$\boxed{n \geq 96.04}$$

CLT much better than Chebyshev.

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4. CLT: X_n i.i.d. $\implies \frac{A_n - \mu}{\sigma/\sqrt{n}} \rightarrow \mathcal{N}(0,1)$

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