1. Why counting?
2. Ordered sampling with replacement
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Why counting?

Preview: If you have a finite set $S$ of equally likely outcomes, the probability of an event $A$ is given by:

$$P(A) = \frac{|A|}{|S|}$$

Example: Fair coin flip

$A = \text{coin lands heads}$

$A = \{H\}$

$S = \{H, T\}$

$\Rightarrow P(A) = \frac{|A|}{|S|} = \frac{1}{2}$

Need to be able to count the number of elements in a set in an efficient manner.
Consider a set $S$ with $n$ elements, we want to draw $k$ such that order matters and repetition is allowed.

```
choices: n n n n n n

k
```

$\rightarrow$ total number of ways is $n^k$

In general, First Rule of Counting:

If $k$ choices in succession, where $n_1$ options for first choice, and for each first choice you have $n_2$ options for second choice, and so on, then the total number of ways to make the $k$ choices is

$n_1 \times n_2 \times \ldots \times n_k$. 

Example

$S = \{1, 2, 3\}$

How many ways to choose 2 elements, with replacement and order matters?

$$\begin{array}{ccc}
1 & 1 \\
1 & 2 \\
1 & 3 \\
2 & 1 \\
2 & 2 \\
2 & 3 \\
3 & 1 \\
3 & 2 \\
3 & 3 \\
\end{array}$$

$3 \cdot 3 = 9 \text{ ways}$

$n = 3$

$k = 2$
Ordered Sampling Without Replacement

Consider the same setting as in (2), but without replacement.

Using the first rule of counting, \( \binom{n}{0} \) choices \( n \), \( n-1 \), \( n-2 \), ..., \( n-k+1 \)

\[ k \]

Total number of ways is

\[ n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = n^P_k \]

"n permute k"

\[ n^P_k = \frac{n!}{(n-k)!} \]

\[ \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \cdot (n-k) \cdot (n-k-1) \ldots}{(n-k) \cdot (n-k-1) \ldots} \]
Example

$S = \{1, 2, 3\}$

How many ways are there to choose 2 elements without replacement and order matters?

\[
\begin{array}{c}
  \frac{1}{2} \\
  \frac{1}{3} \\
  \frac{2}{1} \\
  \frac{2}{3} \\
  \frac{3}{1} \\
  \frac{3}{2}
\end{array}
\]

$\Rightarrow 6$ total ways

\[
\frac{n!}{(n-k)!} = \frac{3!}{(3-2)!} = \frac{6}{1} = 6
\]
4) Unordered sampling without replacement

Consider a set $S$ with $n$ elements, we want to draw $k$ such that order does not matter, and no replacement.

**Second Rule of Counting**

\[
\text{# of ways to choose} \quad = \quad \frac{\text{# of ways to choose with order mattering}}{\text{# of ordered ways per unordered way}}
\]

So, \( \frac{n!}{(n-k)!} \) gives $k$ elements but order matters.

There are $k!$ ways to rearrange the $k$ elements.

So, unordered sampling w/o replacement

\[
\Rightarrow \text{total number of ways is} \quad \frac{n!}{k! (n-k)!}
\]

Note: \( \binom{n}{k} = \frac{n!}{(n-k)! (k)!} \)

"n choose k"
Example

\[ S = \{1, 2, 3\} \]

How many ways are there to choose 2 elements without replacement & order does not matter?

\[ \binom{3}{2} = \frac{3!}{(3-2)!} = 3 \]

\[ \binom{1}{2} \quad \binom{2}{1} \quad \binom{2}{3} \quad \binom{3}{1} \quad \binom{3}{2} \]

\[ \{1, 2\} \quad \{1, 3\} \quad \{2, 3\} \]

\[ \frac{2!}{1!} \text{ ordered ways per 1 unordered way} \]

\[ \binom{n}{k} = \binom{3}{2} = \frac{3!}{(3-2)!2!} = \frac{3}{1 \cdot 2!} = 3 \]

\[ n! \]

\[ P_k = \frac{n^k}{k!} \]
Unordered sampling with replacement.

Consider a set $S$ with $n$ elements, we want to draw $k$ such that order does not matter and repetition is allowed.

Can try to use second rule of counting but run into issue of non-constant number of ordered ways per unordered way.

Example:

$S = \{1, 2, 3\}$

1 → apples
2 → bananas
3 → oranges

unlimited supply

how many ways to select 5 pieces of fruit?

Can try second rule of counting:
Pretend order matters
choices: \( \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{5} = 3 \)

Then how many ordered ways for each unordered?

Ex: All bananas: 22222 \( \rightarrow \) one way

1 apple, rest bananas: 12222 \( \{ 
\begin{align*}
21222 \\
22122 \\
22212 \\
22221 \\
\end{align*} \)

\( \Rightarrow \) No easy way to map from ordered to unordered (no constant)

Bins perspective

1 0 0 0 1 0 1
apple banana orange

\( \uparrow \) represent as a binary string

0001010

"Stars and bars"
How many such binary strings?
3 bins $\rightarrow$ 2 bars / dividers
5 fruits $\rightarrow$ 5 stars / balls

\[
\binom{7}{5} = \binom{7+2}{2} = \binom{5+2}{5} = \frac{7!}{(7-5)!5!} = 21\]

Choose which slots to put the bars
Choose which slots to put the stars

In general, \[\binom{k+n-1}{k}\] ways

\[k = \# \text{ choices / fruits}\]
\[n = \# \text{ possibilities for each choice}\]

**Zeroth rule of Counting**

Want to count size of $A$?
Find bijection between $A$ & $B$
where $B$ is easier to count, and
just count size of $B$.
This works since $|A| = |B|$
Summary of Approaches

Number of ways to choose k items from set of size n

<table>
<thead>
<tr>
<th>replacement</th>
<th>no replacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>order matters</td>
<td>n^k</td>
</tr>
<tr>
<td>order doesn't matter</td>
<td>(n+k-1\choose k)</td>
</tr>
</tbody>
</table>

\[
\frac{(n+k-1)!}{(n+k-1-k)! \cdot k!}
\]

Reminder:
Proctoring Gradecape Assignment HW2 Q1

Break: 4:00 PM
Combiningorial Proofs

Idea: Use intuitive counting arguments instead of tedious algebraic manipulation to prove some identity.

Example

Prove \( \binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \ldots + \binom{k}{k} \)

Proof:

LHS: # of ways to choose \( k+1 \) element subset \( X \) from a set \( S \) containing \( n \) elements \( (S = \{1, 2, \ldots, n\}) \)

RHS: Start by cases based on the lowest numbered element of \( X \).

If lowest is 1, we choose \( k \) from the remaining \( n-1 \) elements.

If lowest is 2, we choose \( k \) from the remaining \( n-2 \) elements.

(Continued on the next page.)
the remaining \( n-2 \) elements of \( S \).

Proceeding in this way by cases depending on the first lowest numbered element \( x \) that we select, we sum the ways for each case,

\[
\binom{n-1}{k} + \binom{n-2}{k} + \ldots + \binom{k}{k}
\]

This gives us all the ways to choose \( k \) elements from set of \( n \).

Since LHS and RHS count the same thing but in different ways, the two sides are equal.
Example

Prove \( (\binom{n}{0}) + (\binom{n}{1}) + \ldots + (\binom{n}{n}) = 2^n \)

Proof:

Say you are constructing a binary string that is \( n \) bits long:

**LHS:**

You could make a string with no zeros.

\( L, (\binom{n}{0}) \) ways

You could make a string with one zero.

\( L, (\binom{n}{1}) \) ways

Total: \( (\binom{n}{0}) + (\binom{n}{1}) + \ldots + (\binom{n}{n}) \) ways

to make an \( n \)-bit binary string.

**RHS:**

Two choices for each position/slot/bit of the binary string, zero or one.
\[ 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 \underbrace{}_{n \text{ times}} \Rightarrow 2^n \text{ binary strings} \]

\[
\begin{array}{c}
n = 3 \\
k = 1
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 0 \\
1 & 1 & 0
\end{array}
\]

\[
(\binom{n}{k}) = (\binom{3}{1}) = 3
\]