Today.

Couple of more induction proofs.

Stable Marriage.

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$$\sum_{i=1}^{k+1} \frac{1}{i^2}$$
$$= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.$$
$$\leq 2 + \frac{1}{(k+1)^2}$$

Uh oh?

Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by $\frac{1}{(k+1)^2}$ for S_k.

"$S_k \leq 2 - \frac{1}{(k+1)^2}$" $\implies$ "$S_{k+1} \leq 2$"
Induction step works! No! Not the same statement!!!!
Need to prove "$S_{k+1} \leq 2 - \frac{1}{(k+2)^2}$".

Darn!!!

Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:

Ind hyp: $P(k)$ — "$S_k \leq 2 - f(k)$"

Prove: $P(k+1)$ – "$S_{k+1} \leq 2 - f(k+1)$"

$$S(k+1) = S_k + \frac{1}{(k+1)^2}$$
$$\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}$$

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.

$$\implies S(k+1) \leq 2 - f(k+1).$$

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

$$\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2}?$$

$$1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by } k+1.$$
$$1 \leq 1 + \left(\frac{1}{k} - \frac{1}{k+1}\right) \quad \text{Some math. So yes!}$$

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Stable Marriage Problem

- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of n boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

Example: Brad and Angelina are a rogue couple in S.

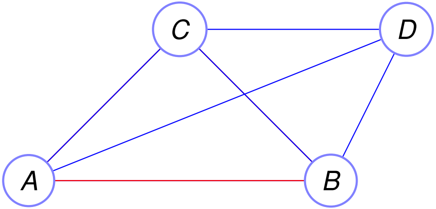
A stable pairing??

Given a set of preferences.

Is there a stable pairing?
How does one find it?

Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer
   (whom she puts on a string.)
3. Rejected boy crosses rejecting girl off his list.

Stop when each girl gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

Boys			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Termination.

Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list. n^2

Terminates in at most $n^2 + 1$ steps!

It gets better every day for girls..

Improvement Lemma: It just gets better for girls.
If on day t a girl g has a boy b on a string,
any boy, b', on g's string for any day $t' > t$
is at least as good as b.

Proof:
$P(k)$- - "boy on g's string is at least as good as b on day $t+k$"

$P(0)$– true. Girl has b on string.

Assume $P(k)$. Let b' be boy **on string** on day $t+k$.

On day $t+k+1$, boy b' comes back.
Girl can choose b', or do better with another boy, b''

That is, $b \leq b'$ by induction hypothesis.
And b'' is better than b' by algorithm.
$\implies$ Girl does at least as well as with b.

$P(k) \implies P(k+1)$. And by principle of induction. $\square$

Pairing when done.

Lemma: Every boy is matched at end.

Proof:
If not, a boy b must have been rejected n times.

Every girl has been proposed to by b,
and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy is on at most one string.

n girls and n boys. Same number of each.

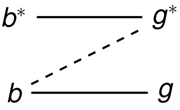
$\implies$ b must be on some girl's string!

Contradiction. $\square$

Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof:
Assume there is a rogue couple; (b, g^*)



b likes g^* more than g.

g^* likes b more than b^*.

Boy b proposes to g^* before proposing to g.

So g^* rejected b (since he moved on)

By improvement lemma, g^* likes b^* better than b.

Contradiction! $\square$

Good for boys? girls?

Is the TMA better for boys? for girls?

Definition: A **pairing is x-optimal** if $x's$ partner
is its best partner in any stable pairing.

Definition: A **pairing is x-pessimal** if $x's$ partner
is its worst partner in any stable pairing.

Definition: A **pairing is boy optimal** if it is x-optimal for **all** boys x.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing.
As well as you can be in a globally stable solution!

Question: Is there a boy or girl optimal pairing?
Is it possible:
b-optimal pairing different from the b'-optimal pairing!
Yes? No?

TMA is optimal!

For boys? For girls?

Theorem: TMA produces a boy-optimal pairing.

Proof:
Assume not: there are boys who do not get their optimal girl.

Let t be first day a boy b gets rejected
by his optimal girl g who he is paired with
in stable pairing S.

b^* - knocks b off of g's string on day $t \implies g$ prefers b^* to b

By choice of t, b^* prefers g to optimal girl.

$\implies b^*$ prefers g to his partner g^* in S.

Rogue couple for S.
So S is not a stable pairing. Contradiction. $\square$

Notes: S - stable. $(b^*, g^*) \in S$. But (b^*, g) is rogue couple!

Used Well-Ordering principle...Induction.

How about for girls?

Theorem: TMA produces girl-pessimal pairing.

T – pairing produced by TMA.

S – worse stable pairing for girl g.

In T, (g, b) is pair.

In S, (g, b^*) is pair.

g likes b^* less than she likes b.

T is boy optimal, so b likes g more than his partner in S.

(g, b) is Rogue couple for S

S is not stable.

Contradiction. $\square$

Notes: Not really induction.
Structural statement: Boy optimality $\implies$ Girl pessimality.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose. $\Longrightarrow$ optimal for girls.

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

Another variation: couples.

Don't go!

Summary.

Link