Today.

Couple of more induction proofs.

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Stable Marriage.

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Darn!!!

Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.

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Can you?

Subtracting off a quadratically decreasing function every time.

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Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

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$$\begin{aligned} S(k+1) &= S_k + \frac{1}{(k+1)^2} \\ &\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.} \end{aligned}$$

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.

$$\implies S(k+1) \leq 2 - f(k+1).$$

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

$$\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2}?$$

$$1 \leq \frac{k+1}{k} - \frac{1}{k+1}$$

Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Stable Marriage Problem

Stable Marriage Problem

- Small town with n boys and n girls.

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- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.

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- Small town with n boys and n girls.
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Stable Marriage Problem

- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.

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- Maximize number of first choices.

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

The best laid plans..

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Brad prefers Angelina to Jennifer.

The best laid plans..

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Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

So..

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Definition: A **pairing** is disjoint set of n boy-girl pairs.

So..

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Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

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Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

So..

Produce a pairing where there is no running off!

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Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

Example: Brad and Angelina are a rogue couple in S.

A stable pairing??

Given a set of preferences.

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Is there a stable pairing?
How does one find it?

A stable pairing??

Given a set of preferences.

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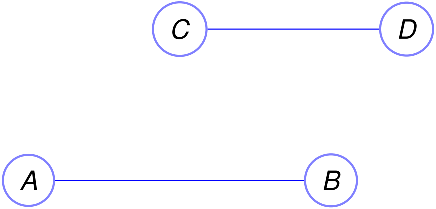
Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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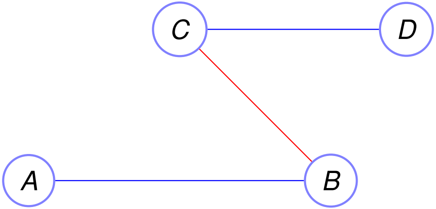

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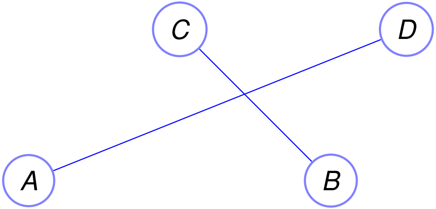

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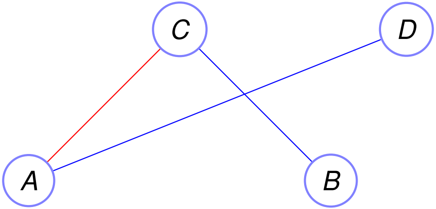

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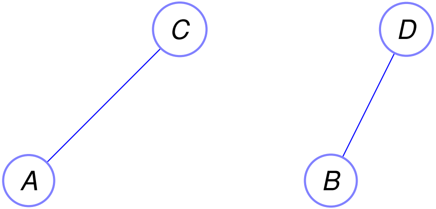

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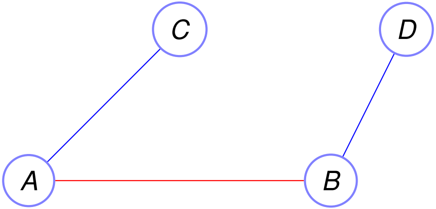

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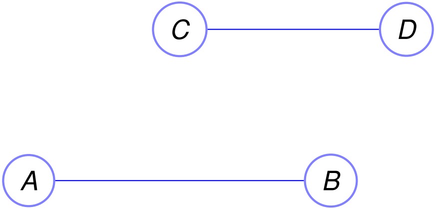

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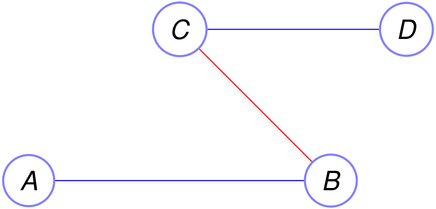

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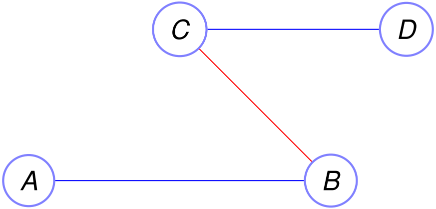

The Traditional Marriage Algorithm.

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Each Day:

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Each Day:

1. Each boy **proposes** to his favorite girl on his list.

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Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer
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3. Rejected boy crosses rejecting girl off his list.

The Traditional Marriage Algorithm.

Each Day:

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2. Each girl rejects all but her favorite proposer (whom she puts on a string.)
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Stop when each girl gets exactly one proposal.

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Each Day:

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Stop when each girl gets exactly one proposal.
Does this terminate?

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Do boys or girls do “better”?

The Traditional Marriage Algorithm.

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Stop when each girl gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

Boys			
A	1	2	3
B	1	2	3
C	2	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

Example.

Boys			
A	1	2	3
B	1	2	3
C	2	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

Example.

Boys			
A	1	2	3
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C	2	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Boys			
A	1	2	3
B	~~1~~	2	3
C	2	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~				
2	C				
3					

Example.

Boys			
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Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, C			
3					

Example.

Boys			
A	1	2	3
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C	~~2~~	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, ~~C~~			
3					

Example.

Boys			
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Girls			
1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	A , C		
2	C	B, ~~C~~	B		
3					

Example.

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A	~~1~~	2	3
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Girls			
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	Day 1	Day 2	Day 3	Day 4	Day 5
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3					

Example.

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Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	
2	C	B, ~~C~~	B	A,B	
3					

Example.

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A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

Girls			
1	C	A	B
2	A	B	C
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	
2	C	B, ~~C~~	B	A, ~~B~~	
3					

Example.

Boys			
A	~~1~~	2	3
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C	~~2~~	1	3

Girls			
1	C	A	B
2	A	B	C
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Example.

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A	~~1~~	2	3
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Termination.

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Every non-terminated day a boy **crossed** an item off the list.

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Every non-terminated day a boy **crossed** an item off the list.

Total size of lists?

Termination.

Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list.

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Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list. n^2

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Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list. n^2

Terminates in at most $n^2 + 1$ steps!

It gets better every day for girls..

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Improvement Lemma: It just gets better for girls.

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If on day t a girl g has a boy b on a string,

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If on day t a girl g has a boy b on a string,
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If on day t a girl g has a boy b on a string,
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It gets better every day for girls..

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If on day t a girl g has a boy b on a string,
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Proof:
$P(k)$- - "boy on g's string is at least as good as b on day $t+k$"

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$P(0)$– true. Girl has b on string.

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Proof:

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Assume $P(k)$. Let b' be boy **on string** on day $t+k$.

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$P(0)$– true. Girl has b on string.

Assume $P(k)$. Let b' be boy **on string** on day $t+k$.

On day $t+k+1$, boy b' comes back.

It gets better every day for girls..

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If on day t a girl g has a boy b on a string,
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On day $t+k+1$, boy b' comes back.
Girl can choose b',

It gets better every day for girls..

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$P(0)$– true. Girl has b on string.

Assume $P(k)$. Let b' be boy **on string** on day $t+k$.

On day $t+k+1$, boy b' comes back.
Girl can choose b', or do better with another boy, b''

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That is,

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On day $t+k+1$, boy b' comes back.
Girl can choose b', or do better with another boy, b''

That is, $b \leq b'$ by induction hypothesis.

It gets better every day for girls..

Improvement Lemma: It just gets better for girls.
If on day t a girl g has a boy b on a string,
any boy, b', on g's string for any day $t' > t$
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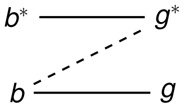

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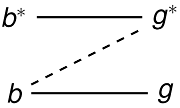

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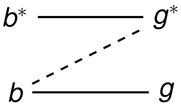

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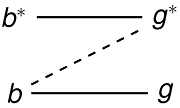

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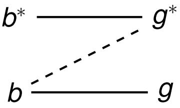

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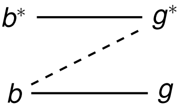

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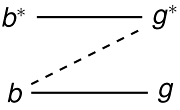

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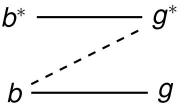

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Theorem: TMA produces a boy-optimal pairing.

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For boys? For girls?

Theorem: TMA produces a boy-optimal pairing.

Proof:
Assume not: there are boys who do not get their optimal girl.

Let t be first day a boy b gets rejected
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Quick Questions.

How does one make it better for girls?

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SMA - stable marriage algorithm. One side proposes.

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Girls could propose.

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Girls could propose. $\Longrightarrow$ optimal for girls.

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The method was used to match residents to hospitals.

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▸ Link