Today.
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Comment: Add 0.
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Comment: Add 0. Poll. Proof that $3 | n^3 - n$. 
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Add $(k - k)$. 
Today.

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Induction: Some quibbles.
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Add $(k - k)$.

Induction: Some quibbles.

What did you learn in 61A?
Comment: Add 0. Poll. Proof that $3 | n^3 - n$.

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Induction: Some quibbles.

What did you learn in 61A?

Induction and Recursion
Comment: Add 0. Poll. Proof that $3 | n^3 - n$.
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Induction: Some quibbles.
What did you learn in 61A?
Induction and Recursion
Couple of more induction proofs.
Comment: Add 0. Poll. Proof that $3 | n^3 - n$.
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Induction: Some quibbles.
What did you learn in 61A?
Induction and Recursion
Couple of more induction proofs.
Stable Marriage.
Some quibbles.

The induction principle works on the natural numbers.
Some quibbles.

The induction principle works on the natural numbers. Proves statements of form: $\forall n \in \mathbb{N}, P(n)$. 

Yes. What if the statement is only for $n \geq 3$?

Restate as: $\forall n \in \mathbb{N}, Q(n)$ where $Q(n) = \text{"}(n \geq 3) \Rightarrow P(n)\"$.

Base Case: typically start at 3. Since $\forall n \in \mathbb{N}, Q(n) \Rightarrow Q(n+1)$ is trivially true before 3.

Can you do induction over other things? Yes. Any set where any subset of the set has a smallest element. In some sense, the natural numbers.
Some quibbles.

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$\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)$
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Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$. 

Instead of proof, let's write some code!

```python
def find_x_y(n):
    if (n==12): return (3,0)
    elif (n==13): return(2,1)
    elif (n==14): return(1,2)
    elif (n==15): return(0,3)
    else:
        (x',y') = find_x_y(n-4)
        return(x'+1,y')
```

Base cases: $P(12), P(13), P(14), P(15)$.

Yes.

Strong Induction step: Recursive call is correct: $P(n-4) \Rightarrow P(n)$.

$n-4 = 4x' + 5y' \Rightarrow n = 4(x'+1) + 5y'$.

Slight differences: showed for all $n \geq 16$ that $\land P(i) = \Rightarrow P(n)$. 

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Base cases: P(12) , P(13) , P(14)
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n - 4 = 4x' + 5y' \implies n = 4(x' + 1) + 5(y')
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Slight differences: showed for all $n \geq 16$ that $\land_{i=4}^{n-1} P(i) \implies P(n)$. 
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)
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Base: $P(1)$. 

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Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$\sum_{i=1}^{k+1} \frac{1}{i^2}$

Uh oh?

Hmmm...

It better be that any sum is strictly less than $2$.

How much less?

At least by $1 \left( k + 1 \right)$ for $S_k$.

"$S_k \leq 2 - 1 \left( k + 1 \right)$" $\Rightarrow$ "$S_{k+1} \leq 2$".

Induction step works!

No!

Not the same statement!!!!

Need to prove "$S_{k+1} \leq 2 - 1 \left( k + 2 \right)$".

Darn!!!
Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

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\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
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\]

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\leq 2 + \frac{1}{(k+1)^2}.
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\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
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Hmmm... It better be that any sum is strictly less than 2.
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Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

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\]

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\leq 2 + \frac{1}{(k+1)^2}.
\]

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Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by $\frac{1}{(k+1)^2}$ for $S_k$. 
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

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“\( S_k \leq 2 - \frac{1}{(k+1)^2} \)”
Strengthening: need to...

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“\( S_k \leq 2 - \frac{1}{(k+1)^2} \)” \( \implies \) “\( S_{k+1} \leq 2 \)”
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"$S_k \leq 2 - \frac{1}{(k+1)^2}$" $\implies$ "$S_{k+1} \leq 2$"

Induction step works!
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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How much less? At least by $\frac{1}{(k+1)^2}$ for $S_k$.

$“S_k \leq 2 - \frac{1}{(k+1)^2}” \implies “S_{k+1} \leq 2”$

Induction step works! No!
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). \((S_n = \sum_{i=1}^{n} \frac{1}{i^2}.\)

Base: \( P(1) \). \( 1 \leq 2 \).
Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).
\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
\]
\[
\leq 2 + \frac{1}{(k+1)^2}.
\]

Uh oh?

Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

\( "S_k \leq 2 - \frac{1}{(k+1)^2}" \implies "S_{k+1} \leq 2" \)

Induction step works! No! Not the same statement!!!!
Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2. \) \((S_n = \sum_{i=1}^{n} \frac{1}{i^2}).\)

Base: \( P(1). \) \( 1 \leq 2. \)

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2. \)

\[
\begin{align*}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}. \\
&\leq 2 + \frac{1}{(k+1)^2}.
\end{align*}
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k. \)

"\( S_k \leq 2 - \frac{1}{(k+1)^2} \)" \( \implies \) "\( S_{k+1} \leq 2 \)"

Induction step works! No! Not the same statement!!!!

Need to prove "\( S_{k+1} \leq 2 - \frac{1}{(k+2)^2} \)."
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Base: \( P(1) \). \( 1 \leq 2 \).

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).

\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
\]

\[
\leq 2 + \frac{1}{(k+1)^2}
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

\( "S_k \leq 2 - \frac{1}{(k+1)^2}" \) \( \implies "S_{k+1} \leq 2" \)

Induction step works! No! Not the same statement!!!!

Need to prove \( "S_{k+1} \leq 2 - \frac{1}{(k+2)^2}" \).
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Base: \( P(1) \). \( 1 \leq 2 \).

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).
\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 + \frac{1}{(k+1)^2}
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

"\( S_k \leq 2 - \frac{1}{(k+1)^2} \)" \( \implies \) "\( S_{k+1} \leq 2 \)"

Induction step works! No! Not the same statement!!!!

Need to prove "\( S_{k+1} \leq 2 - \frac{1}{(k+2)^2} \)."

Darn!!!
Strengthening: how?
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k)$ – "$S_k \leq 2 - f(k)$"
Prove: $P(k+1)$ – "$S_{k+1} \leq 2 - f(k+1)$"

$S_{k+1} = S_k + \frac{1}{(k+1)^2} \leq 2 - f(k) + \frac{1}{(k+1)^2}$.

By ind. hyp.
Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.

$\Rightarrow S_{k+1} \leq 2 - f(k+1)$.

Can you?
Subtracting off a quadratically decreasing function every time.
Maybe a linearly decreasing function to keep positive?
Try $f(k) = \frac{1}{k^2}$.

$\frac{1}{k^2} \leq \frac{1}{k+1} - \left(\frac{1}{k} - \frac{1}{k+1}\right)$.

$\Rightarrow 1 \leq 1 + \left(\frac{1}{k} - \frac{1}{k+1}\right)$.

Multiplied by $k+1$.

So yes!
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k)$
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
**Strengthening: how?**

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

**Proof:**
Ind hyp: $P(k)$ — “$S_k \leq 2 - f(k)$”
Prove: $P(k + 1)$
Strengthening: how?

Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow “S_k \leq 2 - f(k)”$
Prove: $P(k + 1) \rightarrow “S_{k+1} \leq 2 - f(k + 1)”$
Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\]
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow \text{“} S_k \leq 2 - f(k) \text{“}$
Prove: $P(k + 1) \rightarrow \text{“} S_{k+1} \leq 2 - f(k + 1) \text{“}$

$$S(k + 1) = S_k + \frac{1}{(k+1)^2}$$

$$\leq 2 - f(k) + \frac{1}{(k+1)^2}$$
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k+1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow "S_k \leq 2 - f(k)"
Prove: $P(k + 1) \rightarrow "S_{k+1} \leq 2 - f(k + 1)"

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow "S_k \leq 2 - f(k)"

Prove: $P(k + 1) \rightarrow "S_{k+1} \leq 2 - f(k + 1)"

$S(k + 1) = S_k + \frac{1}{(k+1)^2}$

$\leq 2 - f(k) + \frac{1}{(k+1)^2}$ By ind. hyp.

Choose $f(k + 1) \leq f(k) - \frac{1}{(k+1)^2}$.
Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k+1) \)”

\[
S(k+1) = S_k + \frac{1}{(k+1)^2}
\]
\[
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k+1) \leq f(k) - \frac{1}{(k+1)^2} \).
\[
\Rightarrow S(k+1) \leq 2 - f(k+1).
\]
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} ) \)

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k+1) \)”

\[
S(k+1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k+1) \leq f(k) - \frac{1}{(k+1)^2} \).

\( \implies S(k+1) \leq 2 - f(k+1) \).

Can you?
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow "$S_k \leq 2 - f(k)"
Prove: $P(k+1) \rightarrow "$S_{k+1} \leq 2 - f(k+1)"

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}
\]

Choose $f(k + 1) \leq f(k) - \frac{1}{(k+1)^2}$.

$\Rightarrow S(k + 1) \leq 2 - f(k + 1)$.

Can you?
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2}) \)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\]
\[
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[
\Rightarrow S(k + 1) \leq 2 - f(k + 1).
\]

Can you?
Subtracting off a quadratically decreasing function every time.
Strenthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow \text{"}S_k \leq 2 - f(k)\text{"}$
Prove: $P(k+1) \rightarrow \text{"}S_{k+1} \leq 2 - f(k + 1)\text{"}$

\[ S(k+1) = S_k + \frac{1}{(k+1)^2} \]
\[ \leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.} \]

Choose $f(k + 1) \leq f(k) - \frac{1}{(k+1)^2}$.
\[ \implies S(k+1) \leq 2 - f(k + 1). \]

Can you?
Subtracting off a quadratically decreasing function every time. Maybe a linearly decreasing function to keep positive?
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). \((S_n = \sum_{i=1}^{n} \frac{1}{i^2})\)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k+1) \)”

\[
S(k+1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k+1) \leq f(k) - \frac{1}{(k+1)^2} \).
\[
\Rightarrow S(k+1) \leq 2 - f(k+1).
\]

Can you?
Subtracting off a quadratically decreasing function every time.
Maybe a linearly decreasing function to keep positive?
Try \( f(k) = \frac{1}{k} \)
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} .) \)

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k+1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\]
\[
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[
\implies S(k + 1) \leq 2 - f(k + 1).
\]

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?
\]
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[
\implies S(k + 1) \leq 2 - f(k + 1).
\]

Can you?
Subtracting off a quadratically decreasing function every time. Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?
\]
Strengthening: how?

Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k+1) \)”

\[
S(k+1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}
\]

Choose \( f(k+1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[
\implies S(k+1) \leq 2 - f(k+1).
\]

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} \text{?}
\]

\[
1 \leq \frac{k+1}{k} - \frac{1}{k+1}
\]
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) \((S_n = \sum_{i=1}^{n} \frac{1}{i^2}.)\)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k+1) \)”

\[
S(k+1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k+1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[
\Rightarrow S(k+1) \leq 2 - f(k+1).
\]

Can you?
Subtracting off a quadratically decreasing function every time.
Maybe a linearly decreasing function to keep positive?
Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?
\]

\[
1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by } k+1.
\]

\[
1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right)
\]
**Strengthening: how?**

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

**Proof:**

Ind hyp: $P(k) \rightarrow "S_k \leq 2 - f(k)"

Prove: $P(k + 1) \rightarrow "S_{k+1} \leq 2 - f(k + 1)"$

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}
\]

Choose $f(k + 1) \leq f(k) - \frac{1}{(k+1)^2}.$

\[\implies S(k + 1) \leq 2 - f(k + 1).\]

Can you?

Subtracting off a quadratically decreasing function every time. Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

\[\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?\]

\[1 \leq \frac{k+1}{k} - \frac{1}{k+1} \text{ Multiplied by } k + 1.\]

\[1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right) \text{ Some math.}\]
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k)$ — “$S_k \leq 2 - f(k)$”
Prove: $P(k+1)$ — “$S_{k+1} \leq 2 - f(k+1)$”

\[
S(k+1) = S_k + \frac{1}{(k+1)^2}
\]
\[
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.

\[\implies S(k+1) \leq 2 - f(k+1).\]

Can you?

Subtracting off a quadratically decreasing function every time.
Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

\[\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} \quad ?\]

\[1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by } k+1.\]
\[1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right) \quad \text{Some math. So yes!}\]
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:

Ind hyp: $P(k) — "S_k \leq 2 - f(k)"

Prove: $P(k+1) - "S_{k+1} \leq 2 - f(k+1)"

\[
S(k+1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}
\]

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}.$

\[
\Rightarrow S(k+1) \leq 2 - f(k+1).
\]

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?
\]

\[
1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by } k+1.
\]

\[
1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right) \quad \text{Some math. So yes!}
\]

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$. 
Stable Matching Problem

- There are n candidates and n jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.

How should they be matched?
Stable Matching Problem

- $n$ candidates and $n$ jobs.
Stable Matching Problem

- $n$ candidates and $n$ jobs.
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Stable Matching Problem

- $n$ candidates and $n$ jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.
Stable Matching Problem

- $n$ candidates and $n$ jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.

How should they be matched?
Count the ways..

- Maximize total satisfaction.
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
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Davis prefers the Lakers.
Lakers prefer Davis.
The best laid plans..

Consider the pairs..

▶ (Anthony) Davis and Pelicans
▶ (Lonzo) Ball and Lakers

Davis prefers the Lakers.
Lakers prefer Davis.
Uh..oh.
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
Lakers prefer Davis.
Uh..oh. Sad Lonzo and Pelicans.
So..

Produce a pairing where there are no crazy moves!
So..

Produce a pairing where there are no crazy moves!

**Definition:** A **pairing** is disjoint set of $n$ job-candidate pairs.
So..

Produce a pairing where there are no crazy moves!

**Definition:** A **pairing** is a disjoint set of $n$ job-candidate pairs.

Example: A pairing $S = \{(\text{Lakers, Ball}); (\text{Pelicans, Davis})\}$. 
Produce a pairing where there are no crazy moves!

**Definition:** A **pairing** is disjoint set of $n$ job-candidate pairs.

Example: A pairing $S = \{(Lakers, Ball); (Pelicans, Davis)\}$.

**Definition:** A **rogue couple** $b, g^*$ for a pairing $S$: $b$ and $g^*$ prefer each other to their partners in $S$
Produce a pairing where there are no crazy moves!

**Definition:** A **pairing** is disjoint set of $n$ job-candidate pairs.

Example: A pairing $S = \{(Lakers, Ball); (Pelicans, Davis)\}.$

**Definition:** A **rogue couple** $b, g^*$ for a pairing $S$: $b$ and $g^*$ prefer each other to their partners in $S$

Example: Davis and Lakers are a rogue couple in $S.$
A stable pairing??

Given a set of preferences.
A stable pairing??

Given a set of preferences.
Is there a stable pairing?
How does one find it?
A stable pairing??

Given a set of preferences.

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Consider a single type version: stable roommates.

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- **A** - **B**
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A graph representation is shown with nodes A, B, C, and D, where C and D are connected in one direction, and A and B are connected in another direction.
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A → B
B → C
C → D
D → A
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Given a set of preferences.

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Diagram: 

- A is paired with B
- C is paired with D
A stable pairing??

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A — B
C — D

A and B are paired, as are C and D.
A stable pairing??

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\[\begin{array}{c}
C \\
D
\end{array}\]

\[\begin{array}{c}
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A | C
---|---
B | D

A | B
---|---
C |
The Propose and Reject Algorithm.

Each Day:
1. Each job proposes to its favorite candidate on its list.
2. Each candidate rejects all but their favorite proposer (whom they put on a string.)
3. Rejected job crosses rejecting candidate off its list.

Stop when each job gets exactly one proposal.

Does this terminate?...produce a pairing?....a stable pairing? Do jobs or candidates do "better"?
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Do jobs or candidates do “better”? 

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<th>Candidates</th>
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<tbody>
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<td>B</td>
<td>1 2 3</td>
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<tr>
<td>C</td>
<td>2 1 3</td>
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### Example.

<table>
<thead>
<tr>
<th>Jobs</th>
<th>Candidates</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 2 3</td>
<td>1 C A B</td>
</tr>
<tr>
<td>B 2 3</td>
<td>2 A B C</td>
</tr>
<tr>
<td>C 1 3</td>
<td>3 A C B</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>A</td>
<td>A, C</td>
<td>C</td>
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</tr>
<tr>
<td>2</td>
<td>C</td>
<td>B, C</td>
<td>B</td>
<td>A, B</td>
<td></td>
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<tr>
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<td>A, B</td>
<td>A</td>
<td>A, C</td>
<td>C</td>
<td>A, B</td>
</tr>
<tr>
<td>C</td>
<td>B, C</td>
<td>B</td>
<td>A, B</td>
<td></td>
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Day 1: A, B
Day 2: A
Day 3: A, C
Day 4: C
Day 5: A, B
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<tr>
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<td>X</td>
</tr>
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<td>X</td>
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<td>C</td>
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<td>A, B</td>
</tr>
<tr>
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12 / 24
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</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
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</table>
Termination.
Every non-terminated day a job crossed an item off the list.
Termination.

Every non-terminated day a job crossed an item off the list.  
Total size of lists?
Termination.

Every non-terminated day a job crossed an item off the list.
Total size of lists? $n$ jobs, $n$ length list.
Termination.

Every non-terminated day a job crossed an item off the list.
Total size of lists? $n$ jobs, $n$ length list. $n^2$
Termination.

Every non-terminated day a job **crossed** an item off the list.
Total size of lists? $n$ jobs, $n$ length list. $n^2$
Terminates in $\leq n^2$ steps!
It gets better every day for candidates.

Improvement Lemma: It just gets better for candidates.

If on day $t$, a candidate $g$ has a job $b$ on a string, any job, $b'$, on candidate $g$'s string for any day $t'$ > $t$ is at least as good as $b$.

Example: Candidate "Alice" has job "Amalgamated Concrete" on string on day 5. She has job "Amalgamated Asphalt" on string on day 7. Does Alice prefer "Amalgamated Asphalt" or "Amalgamated Concrete"?

$g$ - 'Alice', $b$ - 'Am. Con.', $b'$ - 'Am. Asph.', $t$ = 5, $t'$ = 7.

Improvement Lemma says she prefers 'Amalgamated Asphalt'.

Day 10: Can Alice have "Amalgamated Asphalt" on her string?

Yes. Alice prefers day 10 job as much as day 7 job. Here, $b = b'$.

Why is lemma true?

Proof Idea: She can always keep the previous job on the string.
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Improvement Lemma

Improvement Lemma: It just gets better for candidates.

Proof:
Let $g$ have a job $b$ on its string on day $t$. Assume $g$ has a job $b'$ on its string on day $t+k$. On day $t+k+1$, job $b'$ comes back. Candidate $g$ can choose $b'$, or do better with another job, $b''$. That is, $b' \leq b$ by induction hypothesis. And $b''$ is better than $b'$ by algorithm. $\Rightarrow$ Candidate does at least as well as with $b''$. $P(k) = \Rightarrow P(k+1)$.

And by principle of induction, lemma holds for every day after $t$. 


Improvement Lemma

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If on day $t$ a candidate $g$ has a job $b$ on a string, any job, $b'$, on $g$'s string for any day $t' > t$ is at least as good as $b$. 

**Proof:**

$P(k) - \text{“job on } g \text{’s string is at least as good as } b \text{ on day } t + k \text{”}$

$P(0)$ – true. Candidate has $b$ on string.

Assume $P(k)$. Let $b'$ be job on string on day $t + k$. On day $t + k + 1$, job $b'$ comes back. Candidate $g$ can choose $b'$, or do better with another job, $b''$. That is, $b' \leq b$ by induction hypothesis. And $b''$ is better than $b'$ by algorithm. $\Rightarrow$ Candidate does at least as well as with $b$. 

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$P(k)$ - “job on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$ - true. Candidate has $b$ on string.

Assume $P(k)$. Let $b'$ be job **on string** on day $t + k$.

On day $t + k + 1$, job $b'$ comes back.

Candidate $g$ can choose $b'$, or do better with another job, $b''$

That is, $b' \leq b$ by induction hypothesis.

And $b''$ is better than $b'$ by algorithm.

$\implies$ Candidate does at least as well as with $b$. 
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**Lemma:** Every job is matched at end.
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**Proof:**
Assume there is a rogue couple; \((b, g^*)\)

\(b\) proposes to \(g^*\) before proposing to \(g\).

So \(g^*\) rejected \(b\) (since he moved on).

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\begin{array}{ccc}
  b^* & \longrightarrow & g^* \\
  b & \longleftarrow & g \\
\end{array}
\]

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Good for jobs? candidates?

Is the Job-Proposes better for jobs?
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Is the Job-Proposes better for jobs? for candidates?

Definition:
A pairing is $x$-optimal if $x'$'s partner is its best partner in any stable pairing.

Definition:
A pairing is $x$-pessimal if $x'$'s partner is its worst partner in any stable pairing.

Definition:
A pairing is job optimal if it is $x$-optimal for all jobs $x$.

..and so on for job pessimal, candidate optimal, candidate pessimal.

Claim: The optimal partner for a job must be first in its preference list.

True? False!

Subtlety here: Best partner in any stable pairing. As well as you can be in a globally stable solution!

Question: Is there a job or candidate optimal pairing?

Is it possible: $b$-optimal pairing different from the $b'$-optimal pairing!

Yes? No?
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Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).
Stable? Yes.
Optimal for B? Notice: only one stable pairing. So this is the best B can do in a stable pairing. So optimal for B.
Also optimal for A, 1 and 2.
Also pessimal for A, B, 1 and 2.

Pairing S: (A, 1), (B, 2).
Stable? Yes.
Pairing T: (A, 2), (B, 1).
Also Stable.
Which is optimal for A? S
Which is optimal for B? S
Which is optimal for 1? T
Which is optimal for 2? T
Understanding Optimality: by example.

<table>
<thead>
<tr>
<th></th>
<th>1,2</th>
<th>1:</th>
<th>2:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1,2</td>
<td>A,B</td>
<td></td>
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B: 2,1  2: A,B
Understanding Optimality: by example.

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B:  1,2

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Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2  1: B,A
B: 2,1  2: A,B

Pairing S: (A, 1), (B, 2). Stable?
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
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Pairing S: (A, 1), (B, 2).  Stable? Yes.
Understanding Optimality: by example.

A: 1,2  
B: 1,2  

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
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Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2  
B: 2,1  

Pairing S: (A, 1), (B, 2). Stable? Yes.
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: \((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?
Notice: only one stable pairing.
So this is the best \(B\) can do in a stable pairing.
So optimal for \(B\).

Also optimal for \(A\), 1 and 2. Also pessimal for \(A, B, 1\) and 2.

A: 1,2  1: B,A
B: 2,1  2: A,B

Pairing \(S\): \((A, 1), (B, 2)\). Stable? Yes.
Pairing \(T\): \((A, 2), (B, 1)\).
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
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Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.
Understanding Optimality: by example.

\[
\begin{array}{c|c|c}
A: & 1,2 & 1: A,B \\
B: & 1,2 & 2: B,A \\
\end{array}
\]

Consider pairing: \((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?

Notice: only one stable pairing.
So this is the best \(B\) can do in a stable pairing.
So optimal for \(B\).

Also optimal for \(A\), 1 and 2. Also pessimal for \(A,B, 1\) and 2.

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\begin{array}{c|c|c}
A: & 1,2 & 1: B,A \\
B: & 2,1 & 2: A,B \\
\end{array}
\]

Pairing \(S\): \((A, 1), (B, 2)\). Stable? Yes.

Pairing \(T\): \((A, 2), (B, 1)\). Also Stable.

Which is optimal for \(A\)?
Understanding Optimality: by example.

\[
\begin{align*}
\text{A:} & \quad 1,2 \quad 1: \quad A, B \\
\text{B:} & \quad 1,2 \quad 2: \quad B, A
\end{align*}
\]

Consider pairing: \((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?
Notice: only one stable pairing.
So this is the best \(B\) can do in a stable pairing.
So optimal for \(B\).

Also optimal for \(A\), 1 and 2. Also pessimal for \(A, B, 1\) and 2.

\[
\begin{align*}
\text{A:} & \quad 1,2 \quad 1: \quad B, A \\
\text{B:} & \quad 2,1 \quad 2: \quad A, B
\end{align*}
\]

Pairing \(S\): \((A, 1), (B, 2)\). Stable? Yes.

Pairing \(T\): \((A, 2), (B, 1)\). Also Stable.

Which is optimal for \(A\)? \(S\)
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2 1: B,A
B: 2,1 2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S Which is optimal for B?
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
  Notice: only one stable pairing.
  So this is the best B can do in a stable pairing.
  So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2 1: B,A
B: 2,1 2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S  Which is optimal for B? S
Understanding Optimality: by example.

A: 1,2  
1: A,B
B: 1,2  
2: B,A

Consider pairing: \((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?
Notice: only one stable pairing.
So this is the best \(B\) can do in a stable pairing.
So optimal for \(B\).

Also optimal for \(A\), 1 and 2. Also pessimal for \(A,B,1\) and 2.

A: 1,2  
1: B,A
B: 2,1  
2: A,B

Pairing \(S\): \((A, 1), (B, 2)\). Stable? Yes.
Pairing \(T\): \((A, 2), (B, 1)\). Also Stable.

Which is optimal for \(A\)? \(S\)  
Which is optimal for \(B\)? \(S\)  
Which is optimal for 1?
Understanding Optimality: by example.

A: 1,2  
B: 1,2

Consider pairing: (A,1), (B,2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2  
B: 2,1

Pairing S: (A,1), (B,2). Stable? Yes.

Pairing T: (A,2), (B,1). Also Stable.

Which is optimal for A? S  
Which is optimal for B? S  
Which is optimal for 1? T
Understanding Optimality: by example.

A:  1,2  1:  A,B
B:  1,2  2:  B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A:  1,2  1:  B,A
B:  2,1  2:  A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S  Which is optimal for B? S
Which is optimal for 1? T  Which is optimal for 2?
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2 1: B,A
B: 2,1 2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S Which is optimal for B? S
Which is optimal for 1? T Which is optimal for 2? T
Job Propose and Candidate Reject is optimal!

For jobs?

Theorem: Job Propose and Reject produces a job-optimal pairing.

Proof: Assume not: there is a job \( b \) does not get optimal candidate, \( g \).

There is a stable pairing \( S \) where \( b \) and \( g \) are paired.

Let \( t \) be first day job \( b \) gets rejected by its optimal candidate \( g \) who it is paired with in stable pairing \( S \).

\( b^* \) knocks \( b \) off of \( g \)'s string on day \( t \).

\( b^* \) prefers \( g \) to \( b \) by choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.

\( b^* \) prefers \( g \) to its partner \( g^* \) in \( S \).

Rogue couple for \( S \).

So \( S \) is not a stable pairing. Contradiction.

Notes:

\( S \) - stable.

\( (b^*, g^*) \) \( \in \) \( S \).

But \( (b^*, g^*) \) is rogue couple!

Used Well-Ordering principle... Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Theorem:
Job Propose and Reject produces a job-optimal pairing.

Proof:
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ knocks $b$ off of $g$'s string on day $t$.$\Rightarrow g$ prefers $b^*$ to $b$.

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.$\Rightarrow b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing.

Contradiction.

Notes:
$S$ - stable.
$(b^*, g^*) \in S$.
But $(b^*, g)$ is a rogue couple!

Used Well-Ordering principle...

Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not:

**Notes:**
S - stable.

But (b∗, g∗) ∈ S.

But (b∗, g) is a rogue couple!
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

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Assume not: there is a job $b$ does not get optimal candidate, $g$.

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Let $t$ be first day job $b$ gets rejected
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
by its optimal candidate $g$ who it is paired with
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$. 

Notes: $S - \text{stable.}$

$((b^*, g^*) \in S$.

But $(b^*, g^*)$ is rogue couple! 

Used Well-Ordering principle... Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

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Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
   by its optimal candidate $g$ who it is paired with
   in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$'s string on day $t$ $\implies$ $g$ prefers $b^*$ to $b$
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job \( b \) does not get optimal candidate, \( g \).

There is a stable pairing \( S \) where \( b \) and \( g \) are paired.

Let \( t \) be first day job \( b \) gets rejected
  by its optimal candidate \( g \) who it is paired with
  in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

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Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$. 
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

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$b^*$ - knocks $b$ off of $g$’s string on day $t$ $\implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$. 
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

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By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.

\( \implies \) \( b^* \) prefers \( g \) to its partner \( g^* \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
by its optimal candidate $g$ who it is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$ $\implies$ $g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

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by its optimal candidate $g$ who it is paired with
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$\implies$ $b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes:
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

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**Proof:**
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There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
    by its optimal candidate $g$ who it is paired with
    in stable pairing $S$.

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By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction. \[ \Box \]

Notes: $S$ - stable.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

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$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^*, g^*) \in S$. 

Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

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Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

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By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

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Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

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Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$ $\implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!
Used Well-Ordering principle...Induction.
How about for candidates?

Theorem:
Job Propose and Reject produces candidate-pessimal pairing.

$\mathbf{T}$ – pairing produced by JPR.

$\mathbf{S}$ – worse stable pairing for candidate $g$.

In $\mathbf{T}$, $(g, b)$ is pair.

In $\mathbf{S}$, $(g, b^\ast)$ is pair.

$g$ prefers $b$ to $b^\ast$.

$\mathbf{T}$ is job optimal, so $b$ prefers $g$ to its partner in $\mathbf{S}$.

$(g, b)$ is Rogue couple for $\mathbf{S}$.

$\mathbf{S}$ is not stable.

Contradiction.

Notes:
Not really induction.

Structural statement: Job optimality $\implies$ Candidate pessimality.
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

$T$ – pairing produced by JPR.
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

$T$ – pairing produced by JPR.

$S$ – worse stable pairing for candidate $g$. 
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

T – pairing produced by JPR.

S – worse stable pairing for candidate \( g \).

In \( T \), \((g, b)\) is pair.
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

- $T$ – pairing produced by JPR.
- $S$ – worse stable pairing for candidate $g$.

In $T$, $(g, b)$ is pair.
In $S$, $(g, b^*)$ is pair.
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

$T$ – pairing produced by JPR.

$S$ – worse stable pairing for candidate $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ prefers $b$ to $b^*$. 
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

$T$ – pairing produced by JPR.

$S$ – worse stable pairing for candidate $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ prefers $b$ to $b^*$.

$T$ is job optimal, so $b$ prefers $g$ to its partner in $S$. 

How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

$T$ – pairing produced by JPR.

$S$ – worse stable pairing for candidate $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ prefers $b$ to $b^*$.

$T$ is job optimal, so $b$ prefers $g$ to its partner in $S$.

$(g, b)$ is Rogue couple for $S$. 

Notes: Not really induction. Structural statement: Job optimality $\Rightarrow$ Candidate pessimality.
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

$T$ – pairing produced by JPR.
$S$ – worse stable pairing for candidate $g$.

In $T$, $(g, b)$ is pair.
In $S$, $(g, b^*)$ is pair.

$g$ prefers $b$ to $b^*$.

$T$ is job optimal, so $b$ prefers $g$ to its partner in $S$.

$(g, b)$ is Rogue couple for $S$.

$S$ is not stable.
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How does one make it better for candidates?
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Propose and Reject - stable matching algorithm. One side proposes.
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Takeaways.

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