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Today: Finish up induction and start graph theory.

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- $P(x) \implies P(x+1)$ certainly does not hit all of $\mathbb{R}$. Neither does $P(x) \implies P(x+\varepsilon)$ regardless of what ε is.
- Any way of getting to the "next" real number must *not* coincide with our usual notion of an ordering on $\mathbb{R}$.

Well Orderings

Given a set S, a **total ordering** $\leq$ on S is a relation which satisfies, for all $x, y, z \in S$:

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- (Antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$.

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- (Antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$.

Given a set S, a **well ordering**[1] $\leq$ on S is a total ordering that also satisfies the following property:

> ***Well Ordering Property***: *For any non-empty subset $R \subseteq S$, R has a **least element**, that is, an element x such that $x \leq y$ for all $y \in R$.*

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Examples of Orderings

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Subsets of S	Least Element
$\varnothing$	none
$\{x_1\}$	x_1
$\{x_2\}$	x_2
$\{x_3\}$	x_3
$\{x_1, x_2\}$	x_1
$\{x_1, x_3\}$	x_1
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- Specifically, induction on the size of R proves:

$$\forall n \in \mathbb{N} \left[\big((R \subseteq \mathbb{N}) \wedge (|R| = n) \wedge (R \neq \varnothing) \big) \implies Q(R) \right]$$

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- Otherwise, $n+1$ must be the least element of R. $\square$

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- So, $P(n_0 - 1) \implies P(n_0)$ is False, which is a contradiction. $\square$

Well Ordering Principle Conclusions

We can perform induction as long as we have a well ordering.

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- However, the well ordering on $\mathbb{R}$ will be very *bizarre*, so trying to use induction on $\mathbb{R}$ is not very useful.

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Well Ordering Principle Conclusions

We can perform induction as long as we have a well ordering.
A well ordering tells us what the "next" element is.

- Say we want to prove $\forall x \in S,\ P(x)$.
- S has a least element x_0; prove $P(x_0)$.
- Let $R = S \setminus \{x_0\}$; then R has a least element x_1. Prove $P(x_0) \implies P(x_1)$.
- Continue...

So the question is: which sets can be well ordered?

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The Well Ordering Principle can be used instead of induction.

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The Division Algorithm returns $40 = 7 \cdot 5 + 5$.

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- We will skip the proof that q and r are unique.

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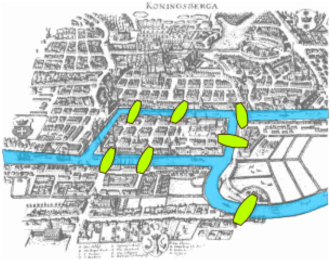
Seven Bridges of Königsberg

New topic: graphs.



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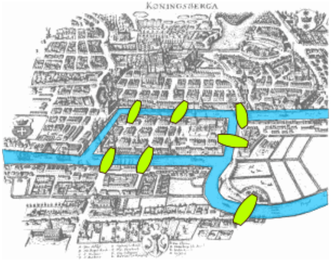

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This problem was solved by Euler in 1736.

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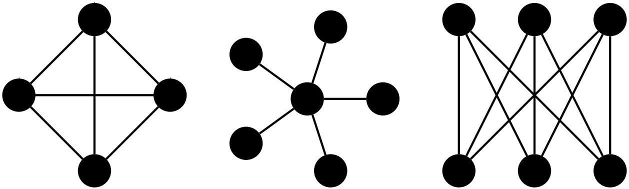

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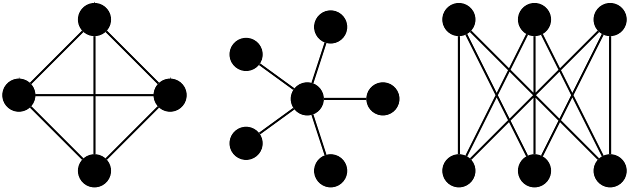

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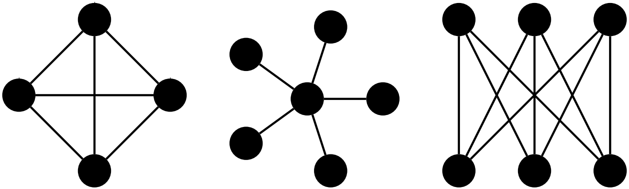

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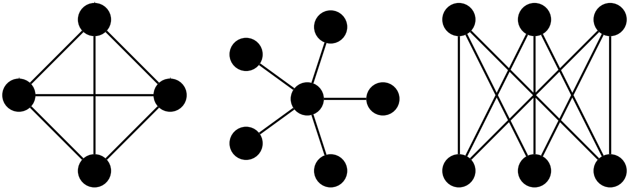

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We only consider finite graphs.

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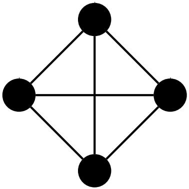

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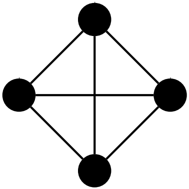

- The **neighbors** of a vertex v are the vertices which are connected (via an edge) to v.

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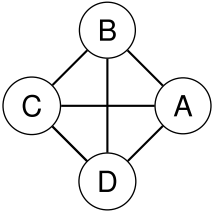
- Total degree is twice the number of handshakes. $\square$

Walks, Paths, Tours, Cycles



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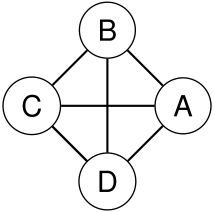

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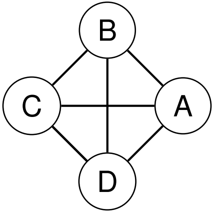

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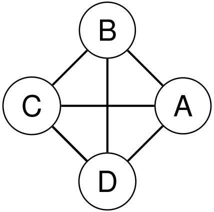

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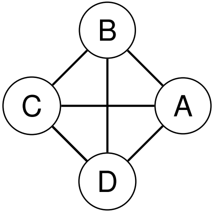

A **walk** is a sequence of edges $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$.

Example: $\{A, B\}, \{B, D\}, \{D, B\}, \{B, C\}$.

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A **tour** is a walk which starts and ends at the same vertex.

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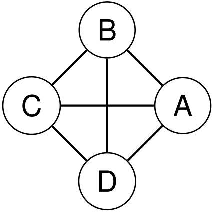

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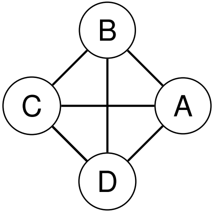

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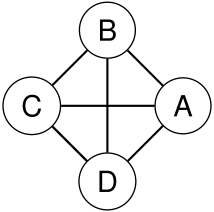

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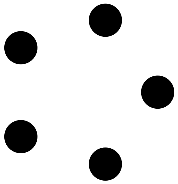

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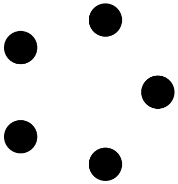

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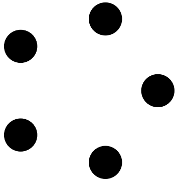

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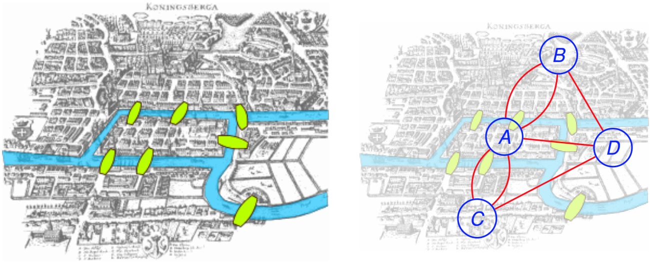
In the directed case, connectivity is not so simple. It may be possible to reach v from u, but not u from v.

The Königsberg Graph



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We *abstract* out the unnecessary details to get a graph.

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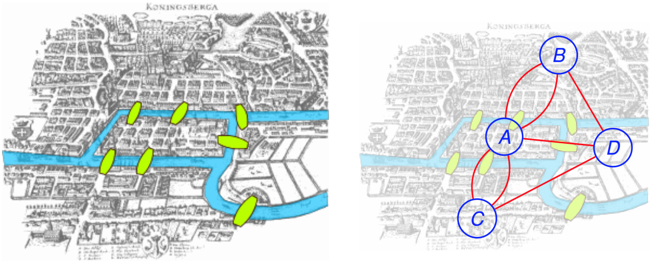

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Königsberg Bridges Problem: Does there exist a tour in the graph which visits every edge exactly once?

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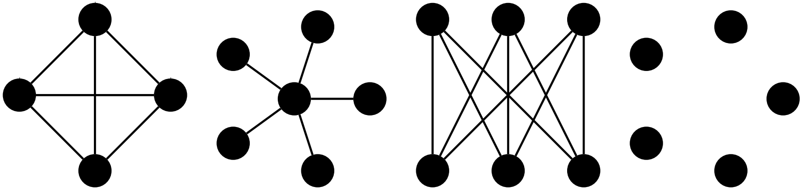
Of the graphs we have seen so far, which have Eulerian tours?



Conditions for Eulerian Tour

Theorem: A graph with no isolated vertices has an Eulerian tour iff it is connected and every vertex has even degree.

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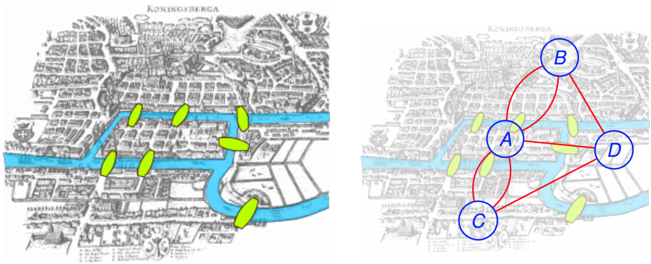
- Each of these components must be connected and each vertex has even degree, so recursively find Eulerian tours.
- The original tour touches each of these Eulerian tours (original graph is connected), so “splice together” the tours. $\square$

Solution to the Königsberg Bridges Problem



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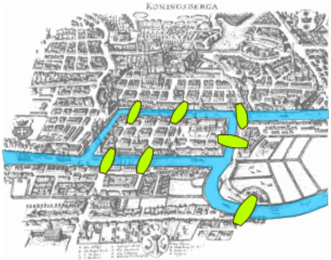

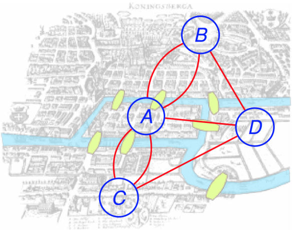

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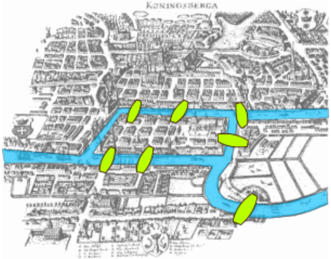

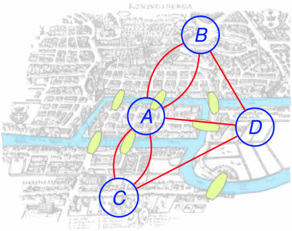

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NO. There is no Eulerian tour!

Summary

Induction:

- Definitions of total ordering and well ordering.
- Well Ordering Principle for $\mathbb{N}$: The usual ordering on $\mathbb{N}$ is a well ordering.
- The Well Ordering Principle is equivalent to induction.
- Green-eyed dragons: common knowledge is the key.

Graph theory:

- Definitions: Graph, vertices, edges, endpoints, incidence, degree, neighbors, isolated vertices, connectedness, walks, paths, tours, cycles...
- Handshaking Lemma
- For graphs without isolated vertices, Eulerian tours exist iff the graph is connected and every vertex has even degree.