Today.

Comment: Add 0.
Today.

Comment: Add 0. Poll.
Today.

Comment: Add 0. Poll.

Add \((k - k)\).
Today.

Comment: Add 0. Poll.

Add \((k - k)\).

Induction: Some quibbles.
Comment: Add 0. Poll.

Add \((k - k)\).

Induction: Some quibbles.

What did you learn in 61A?
Today.

Comment: Add 0. Poll.

Add \((k - k)\).

Induction: Some quibbles.

What did you learn in 61A?

Induction and Recursion
Comment: Add 0. Poll.

Add \((k - k)\).

Induction: Some quibbles.

What did you learn in 61A?

Induction and Recursion

Couple of more induction proofs.
Today.

Comment: Add 0. Poll.

Add \((k - k)\).

Induction: Some quibbles.

What did you learn in 61A?

Induction and Recursion

Couple of more induction proofs.

Stable Marriage.
Some quibbles.

The induction principle works on the natural numbers.
Some quibbles.

The induction principle works on the natural numbers.

Proves statements of form: $\forall n \in \mathbb{N}, P(n)$. 

Yes.

What if the statement is only for $n \geq 3$?

Restate as:

$\forall n \in \mathbb{N}, Q(n)$

where

$Q(n) = (n \geq 3) \implies P(n)$

Base Case: typically start at 3.

Since $\forall n \in \mathbb{N}, Q(n)$ is trivially true before 3.

Can you do induction over other things? Yes.

Any set where any subset of the set has a smallest element.

In some sense, the natural numbers.
Some quibbles.

The induction principle works on the natural numbers. Proves statements of form: \( \forall n \in \mathbb{N}, P(n) \).

Yes.
Some quibbles.

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Proves statements of form: \( \forall \, n \in \mathbb{N}, \, P(n) \).

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What if the statement is only for \( n \geq 3 \)?

\[ \forall \, n \in \mathbb{N}, \, (n \geq 3) \implies P(n) \]
Some quibbles.

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$\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)$

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The induction principle works on the natural numbers. Proves statements of form: $\forall n \in \mathbb{N}, P(n)$.

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$$\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)$$

Restate as:

$$\forall n \in \mathbb{N}, Q(n) \text{ where } Q(n) = "(n \geq 3) \implies P(n)".$$
Some quibbles.

The induction principle works on the natural numbers. Proves statements of form: \( \forall n \in \mathbb{N}, P(n) \).

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Restate as:

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\forall n \in \mathbb{N}, Q(n) \text{ where } Q(n) = "(n \geq 3) \implies P(n)".
\]

Base Case: typically start at 3.
Some quibbles.

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Restate as:

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\forall n \in \mathbb{N}, Q(n) \text{ where } Q(n) = "(n \geq 3) \implies P(n)".
\]

Base Case: typically start at 3.

Since \( \forall n \in \mathbb{N}, Q(n) \implies Q(n+1) \) is trivially true before 3.
Some quibbles.

The induction principle works on the natural numbers.
Proofs statements of form: $\forall n \in \mathbb{N}, P(n)$.

Yes.

What if the statement is only for $n \geq 3$?

$\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)$

Restate as:

$\forall n \in \mathbb{N}, Q(n)$ where $Q(n) = "(n \geq 3) \implies P(n)"$.

Base Case: typically start at 3.
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Some quibbles.

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Yes.

What if the statement is only for $n \geq 3$?

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$$\forall n \in \mathbb{N}, Q(n)$$

where $Q(n) = "(n \geq 3) \implies P(n)"$.

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In some sense, the natural numbers.
Strong Induction and Recursion.

Thm: For every natural number \( n \geq 12 \), \( n = 4x + 5y \).

Instead of proof, let's write some code!

```python
def find_x_y(n):
    if n == 12:
        return (3, 0)
    elif n == 13:
        return (2, 1)
    elif n == 14:
        return (1, 2)
    elif n == 15:
        return (0, 3)
    else:
        (x_prime, y_prime) = find_x_y(n - 4)
        return (x_prime + 1, y_prime)
```

Base cases:
\( P(12), P(13), P(14), P(15) \).

Yes.

Strong Induction step:
Recursive call is correct:
\( P(n-4) \Rightarrow P(n) \).

\( n - 4 = 4x' + 5y' \Rightarrow n = 4(x' + 1) + 5y' \).

Slight differences: showed for all \( n \geq 16 \) that
\( n - 1 = 4P(i) \Rightarrow P(n) \).
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

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    else:
        (x_prime, y_prime) = find_x_y(n - 4)
        return (x_prime + 1, y_prime)
```

Base cases: $P(12)$, $P(13)$, $P(14)$, $P(15)$.

Yes.

Strong Induction step: Recursive call is correct: $P(n - 4) = \Rightarrow P(n)$.

$n - 4 = 4x' + 5y' = \Rightarrow n = 4(x' + 1) + 5y'$.

Slight differences: showed for all $n \geq 16$ that $\land i = 4P(i) = \Rightarrow P(n)$. 
Strong Induction and Recursion.

Thm: For every natural number \( n \geq 12 \), \( n = 4x + 5y \).

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def find_x_y(n):
    if (n == 12) return (3, 0)
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    else:
        (x', y') = find_x_y(n-4)
        return(x'+1, y')
```

Base cases: \( P(12), P(13), P(14), P(15) \).

Yes.

Strong Induction step: Recursive call is correct: \( P(n-4) \Rightarrow P(n) \).

\( n-4 = 4x' + 5y' \Rightarrow n = 4(x'+1) + 5y' \)

Slight differences: showed for all \( n \geq 16 \) that \( \land \sum_{i=1}^{n-1} P(i) \Rightarrow P(n) \).
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

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def find-x-y(n):
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Base cases:
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    elif (n==15): return(0,3)
    else:
        (x',y') = find-x-y(n-4)
        return(x'+1,y')
```

Base cases: \( P(12) \)
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let’s write some code!

def find-x-y(n):
    if (n==12) return (3,0)
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    elif (n==14): return(1,2)
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    else:
        (x’,y’) = find-x-y(n-4)
        return(x’+1,y’)

Base cases: P(12) , P(13)
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let’s write some code!

```python
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```

Base cases: $P(12)$, $P(13)$, $P(14)$
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Strong Induction step:
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

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    else:
        (x',y') = find_x_y(n-4)
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```


Strong Induction step:
Recursive call is correct: $P(n - 4)$
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let’s write some code!

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    else:
        (x',y') = find-x-y(n-4)
        return(x'+1,y')
```

Base cases: $P(12), P(13), P(14), P(15)$. Yes.

Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$. 
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let’s write some code!

def find-x-y(n):
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Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$.

$n - 4 = 4x' + 5y' \implies n = 4(x' + 1) + 5(y')$
Strong Induction and Recursion.

Thm: For every natural number \( n \geq 12 \), \( n = 4x + 5y \).

Instead of proof, let's write some code!

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def find-x-y(n):
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Base cases: \( P(12) \), \( P(13) \), \( P(14) \), \( P(15) \). Yes.

Strong Induction step:

Recursive call is correct: \( P(n-4) \implies P(n) \).
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Base cases: P(12), P(13), P(14), P(15). Yes.

Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$.
$n - 4 = 4x' + 5y' \implies n = 4(x' + 1) + 5(y')$

Slight differences: showed for all $n \geq 16$ that $\bigwedge_{i=4}^{n-1} P(i) \implies P(n)$. 
Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. $(S_n = \sum_{i=1}^{n} \frac{1}{i^2} \cdot)$
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. 
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$. 
Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$. 
Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2. \) \((S_n = \sum_{i=1}^{n} \frac{1}{i^2}).\)

Base: \( P(1). \ 1 \leq 2. \)

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2. \)

Induction step works!

Not the same statement!!!!

Need to prove \( S_{k+1} \leq 2 - \frac{1}{(k+2)^2}. \)
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.

Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.$$
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. 1 $\leq$ 2.
Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$.

$\leq 2 + \frac{1}{(k+1)^2}$.
**Theorem:** For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2. \) \((S_n = \sum_{i=1}^{n} \frac{1}{i^2}.)\)

**Base:** \( P(1). \) \( 1 \leq 2. \)

**Ind Step:** \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2. \)

\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
\]

\[
\leq 2 + \frac{1}{(k+1)^2}.
\]

**Uh oh?**
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Base: \( P(1) \). \( 1 \leq 2 \).

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).

\[
\begin{align*}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
&\leq 2 + \frac{1}{(k+1)^2}
\end{align*}
\]

Uh oh?

Hmmm...
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Base: \( P(1) \). \( 1 \leq 2 \).

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).

\[
\begin{align*}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
&\leq 2 + \frac{1}{(k+1)^2}
\end{align*}
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.

Ind Step: $\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2$.

\[
\begin{align*}
\sum_{i=1}^{k+1} \frac{1}{i^2} & = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
& \leq 2 + \frac{1}{(k+1)^2}.
\end{align*}
\]

Uh oh?

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How much less?
Strengthening: need to...

Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2. \) \((S_n = \sum_{i=1}^{n} \frac{1}{i^2}).\)

Base: \( P(1). \) \( 1 \leq 2. \)

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2. \)

\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
\]

\[
\leq 2 + \frac{1}{(k+1)^2}.
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k. \)
Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).

Base: \( P(1) \). \( 1 \leq 2 \).

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).

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\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
\leq 2 + \frac{1}{(k+1)^2}
\]

Uh oh?

Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

"\( S_k \leq 2 - \frac{1}{(k+1)^2} \)"

""Strengthening: need to...""
Strengthening: need to...

Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2. \) \((S_n = \sum_{i=1}^{n} \frac{1}{i^2}.)\)

Base: \( P(1). \) \( 1 \leq 2. \)

Ind Step: \( \sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2. \)

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\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
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\[
\leq 2 + \frac{1}{(k+1)^2}
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k. \)

“\( S_k \leq 2 - \frac{1}{(k+1)^2} \)” \( \implies \) “\( S_{k+1} \leq 2 \)”
Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Base: $P(1)$. $1 \leq 2$.

Ind Step: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$.

\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
\]

\[
\leq 2 + \frac{1}{(k+1)^2}.
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by $\frac{1}{(k+1)^2}$ for $S_k$.

"$S_k \leq 2 - \frac{1}{(k+1)^2}$" $\implies$ "$S_{k+1} \leq 2$"

Induction step works!
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). \( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).

Base: \( P(1) \). \( 1 \leq 2 \).
Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).

\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
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Hmmm... It better be that any sum is *strictly* less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

“\( S_k \leq 2 - \frac{1}{(k+1)^2} \)” \( \implies \) “\( S_{k+1} \leq 2 \)”

Induction step works! No!
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} .)\)

Base: \( P(1) \). \( 1 \leq 2 \).

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).

\[
\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}.
\]

\[
\leq 2 + \frac{1}{(k+1)^2}.
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

\[“S_k \leq 2 - \frac{1}{(k+1)^2}” \implies “S_{k+1} \leq 2”\]

Induction step works! No! Not the same statement!!!!
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2}. \)

Base: \( P(1) \). \( 1 \leq 2 \).
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Induction step works! No! Not the same statement!!!!

Need to prove “\( S_{k+1} \leq 2 - \frac{1}{(k+2)^2} \)”.
Strengthening: need to...

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

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Need to prove “\( S_{k+1} \leq 2 - \frac{1}{(k+2)^2} \)”.
Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2}. \)

Base: \( P(1) \). \( 1 \leq 2 \).

Ind Step: \( \sum_{i=1}^{k} \frac{1}{i^2} \leq 2 \).

\[
\begin{align*}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \cdot \\
&\leq 2 + \frac{1}{(k+1)^2} \\
\end{align*}
\]

Uh oh?

Hmmm... It better be that any sum is strictly less than 2.

How much less? At least by \( \frac{1}{(k+1)^2} \) for \( S_k \).

“\( S_k \leq 2 - \frac{1}{(k+1)^2} \)” \( \Rightarrow \) “\( S_{k+1} \leq 2 \)”

Induction step works! No! Not the same statement!!!!

Need to prove “\( S_{k+1} \leq 2 - \frac{1}{(k+2)^2} \)”.

Darn!!!
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \)
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Strengthening: how?

Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k + 1) \)
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow "S_k \leq 2 - f(k)"
Prove: $P(k + 1) \rightarrow "S_{k+1} \leq 2 - f(k + 1)"$
Strengthening: how?
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k)$ — “$S_k \leq 2 - f(k)$”
Prove: $P(k + 1)$ — “$S_{k+1} \leq 2 - f(k + 1)$”
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Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”
Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\]
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2}
\]
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) — “S_k \leq 2 - f(k)”$
Prove: $P(k + 1) – “S_{k+1} \leq 2 - f(k + 1)”$

$S(k + 1) = S_k + \frac{1}{(k+1)^2}$

\[ \leq 2 - f(k) + \frac{1}{(k+1)^2} \] By ind. hyp.
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n).$ ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) — “S_k \leq 2 - f(k)”$
Prove: $P(k + 1) — “S_{k+1} \leq 2 - f(k + 1)”$

$$S(k + 1) = S_k + \frac{1}{(k+1)^2}$$

$$\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}$$
Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k+1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow "S_k \leq 2 - f(k)"
Prove: $P(k+1) \rightarrow "S_{k+1} \leq 2 - f(k+1)"

\[ S(k+1) = S_k + \frac{1}{(k+1)^2} \]
\[ \leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.} \]

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.  
\[ \Rightarrow S(k+1) \leq 2 - f(k+1). \]
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:

Ind hyp: $P(k) \rightarrow \text{"} S_k \leq 2 - f(k) \text{"}$

Prove: $P(k+1) \rightarrow \text{"} S_{k+1} \leq 2 - f(k + 1) \text{"}$

\[ S(k + 1) = S_k + \frac{1}{(k+1)^2} \]

\[ \leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.} \]

Choose $f(k + 1) \leq f(k) - \frac{1}{(k+1)^2}$. 

\[ \implies S(k + 1) \leq 2 - f(k + 1). \]

Can you?
Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:

Ind hyp: $P(k) \rightarrow "S_k \leq 2 - f(k)"

Prove: $P(k+1) \rightarrow "S_{k+1} \leq 2 - f(k+1)"

$$S(k + 1) = S_k + \frac{1}{(k+1)^2} \leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}$$

Choose $f(k+1) \leq f(k) - \frac{1}{(k+1)^2}$.

$$\implies S(k + 1) \leq 2 - f(k + 1).$$

Can you?
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2}. \)

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[
\implies S(k + 1) \leq 2 - f(k + 1).
\]

Can you?

Subtracting off a quadratically decreasing function every time.
Strengthening: how?

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

Proof:
Ind hyp: $P(k) \rightarrow \ "S_k \leq 2 - f(k)"$
Prove: $P(k + 1) \rightarrow \ "S_{k+1} \leq 2 - f(k + 1)"$

$S(k + 1) = S_k + \frac{1}{(k+1)^2}$

$\leq 2 - f(k) + \frac{1}{(k+1)^2}$ By ind. hyp.

Choose $f(k + 1) \leq f(k) - \frac{1}{(k+1)^2}$.

$\implies S(k+1) \leq 2 - f(k + 1)$.

Can you?
Subtracting off a quadratically decreasing function every time.
Maybe a linearly decreasing function to keep positive?
Strengthening: how?

Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:
Ind hyp: \( P(k) \rightarrow \text{“} S_k \leq 2 - f(k) \text{“} \)
Prove: \( P(k+1) \rightarrow \text{“} S_{k+1} \leq 2 - f(k+1) \text{“} \)

\[
\begin{align*}
S(k+1) &= S_k + \frac{1}{(k+1)^2} \\
&\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}
\end{align*}
\]

Choose \( f(k+1) \leq f(k) - \frac{1}{(k+1)^2}. \)

\[
\implies S(k+1) \leq 2 - f(k+1).
\]

Can you?
- Subtracting off a quadratically decreasing function every time.
- Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)
Strengthening: how?

Theorem: For all \( n \geq 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n). \) (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:

Ind hyp: \( P(k) \rightarrow \text{“} S_k \leq 2 - f(k) \text{“} \)

Prove: \( P(k+1) \rightarrow \text{“} S_{k+1} \leq 2 - f(k+1) \text{“} \)

\[
S(k+1) = S_k + \frac{1}{(k+1)^2}
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
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Choose \( f(k+1) \leq f(k) - \frac{1}{(k+1)^2}. \)

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\implies S(k+1) \leq 2 - f(k+1).
\]

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} \ ?
\]
**Strenthening: how?**

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

**Proof:**

Ind hyp: $P(k) \rightarrow "S_k \leq 2 - f(k)"$

Prove: $P(k + 1) \rightarrow "S_{k+1} \leq 2 - f(k + 1)"

$$S(k + 1) = S_k + \frac{1}{(k+1)^2} \leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}$$

Choose $f(k + 1) \leq f(k) - \frac{1}{(k+1)^2}$. 

$\implies S(k + 1) \leq 2 - f(k + 1)$. 

Can you?

Subtracting off a quadratically decreasing function every time. Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

$$\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?$$
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[
\Rightarrow S(k + 1) \leq 2 - f(k + 1).
\]

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?
\]

\[
1 \leq \frac{k+1}{k} - \frac{1}{k+1}
\]
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

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S(k + 1) = S_k + \frac{1}{(k+1)^2} \\
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\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[ \implies S(k + 1) \leq 2 - f(k + 1). \]

Can you?

Subtracting off a quadratically decreasing function every time.
Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?
\]

\[
1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by} \ k + 1.
\]

\[
1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right)
\]
**Strengthening: how?**

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n)$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

**Proof:**

Ind hyp: $P(k) \rightarrow "S_k \leq 2 - f(k)"

Prove: $P(k + 1) \rightarrow "S_{k+1} \leq 2 - f(k + 1)"

$S(k + 1) = S_k + \frac{1}{(k+1)^2} 
\leq 2 - f(k) + \frac{1}{(k+1)^2}$ By ind. hyp.

Choose $f(k + 1) \leq f(k) - \frac{1}{(k+1)^2}$.

$\implies S(k + 1) \leq 2 - f(k + 1)$.

Can you?

Subtracting off a quadratically decreasing function every time.

Maybe a linearly decreasing function to keep positive?

Try $f(k) = \frac{1}{k}$

$\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2}$ ?

$1 \leq \frac{k+1}{k} - \frac{1}{k+1}$ Multiplied by $k + 1$.

$1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right)$ Some math.
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). (\( S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).)

Proof:

Ind hyp: \( P(k) — "S_k \leq 2 - f(k)" \)

Prove: \( P(k + 1) — "S_{k+1} \leq 2 - f(k + 1)" \)

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2} \\
\leq 2 - f(k) + \frac{1}{(k+1)^2} \text{ By ind. hyp.}
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Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\[ \implies S(k + 1) \leq 2 - f(k + 1). \]

Can you?

Subtracting off a quadratically decreasing function every time.
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Try \( f(k) = \frac{1}{k} \)

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\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} ?
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\[
1 \leq \frac{k+1}{k} - \frac{1}{k+1} \text{ Multiplied by } k + 1.
\]

\[
1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right) \text{ Some math. So yes!}
\]
Strengthening: how?

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - f(n) \). \( (S_n = \sum_{i=1}^{n} \frac{1}{i^2} \).

Proof:

Ind hyp: \( P(k) \) — “\( S_k \leq 2 - f(k) \)”

Prove: \( P(k + 1) \) — “\( S_{k+1} \leq 2 - f(k + 1) \)”

\[
S(k + 1) = S_k + \frac{1}{(k+1)^2}
\]
\[
\leq 2 - f(k) + \frac{1}{(k+1)^2} \quad \text{By ind. hyp.}
\]

Choose \( f(k + 1) \leq f(k) - \frac{1}{(k+1)^2} \).

\( \implies S(k + 1) \leq 2 - f(k + 1) \).

Can you?

Subtracting off a quadratically decreasing function every time.
Maybe a linearly decreasing function to keep positive?

Try \( f(k) = \frac{1}{k} \)

\[
\frac{1}{k+1} \leq \frac{1}{k} - \frac{1}{(k+1)^2} \ ?
\]

\[
1 \leq \frac{k+1}{k} - \frac{1}{k+1} \quad \text{Multiplied by } k + 1.
\]

\[
1 \leq 1 + \left( \frac{1}{k} - \frac{1}{k+1} \right) \quad \text{Some math. So yes!}
\]

Theorem: For all \( n \geq 1 \), \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} \).
Stable Matching Problem

- Two sets of entities, each of size $n$:
  - $n$ candidates
  - $n$ jobs
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.
- How should they be matched?
Stable Matching Problem

- $n$ candidates and $n$ jobs.
Stable Matching Problem

- $n$ candidates and $n$ jobs.
- Each job has a ranked preference list of candidates.
Stable Matching Problem

- $n$ candidates and $n$ jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.
Stable Matching Problem

- $n$ candidates and $n$ jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.

How should they be matched?
Count the ways..

- Maximize total satisfaction.
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
Lakers prefer Davis.
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
Lakers prefer Davis.
Uh..oh.
Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
Lakers prefer Davis.
Uh..oh. Sad Lonzo and Pelicans.
Producing a pairing where there is no crazy moves!

**Definition:**
A pairing is a disjoint set of \( n \) job-candidate pairs.

**Example:** A pairing \( S = \{ (\text{Lakers}, \text{Ball}); (\text{Pelicans}, \text{Davis}) \} \).

**Definition:** A rogue couple \( (b, g^*) \) for a pairing \( S \):
\( b \) and \( g^* \) prefer each other to their partners in \( S \).

**Example:** Davis and Lakers are a rogue couple in \( S \).
Produce a pairing where there is no crazy moves!

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A stable pairing??

Given a set of preferences.
A stable pairing??

Given a set of preferences.

Is there a stable pairing?
How does one find it?
A stable pairing??

Given a set of preferences.

Is there a stable pairing?
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Consider a single type version: stable roommates.

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A —— C

B —— D

A —— B
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A stable pairing is shown in the diagram.
A stable pairing??

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A graph illustrating the stable pairing:

- A is paired with B
- C is paired with D

This graph represents a stable pairing where no one would prefer to switch partners.
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A → B  
B → C  
C → A  
D → B
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A

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D

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A graph is shown with nodes labeled A, B, C, and D, connected as follows:
- A is connected to B.
- C is connected to D.
- B is connected to A.
The Propose and Reject Algorithm.

Each Day:
1. Each job proposes to its favorite candidate on its list.
2. Each candidate rejects all but their favorite proposer (whom they put on a string.)
3. Rejected job crosses rejecting candidate off its list.

Stop when each job gets exactly one proposal.

Does this terminate?...produce a pairing?....a stable pairing?

Do jobs or candidates do “better”?
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Do jobs or candidates do “better”?
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Day 1

Day 2

Day 3

Day 4

Day 5
Example.

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<td>A, B</td>
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<td>3</td>
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Example.

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<th>Jobs</th>
<th>Candidates</th>
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<tr>
<td>A</td>
<td>1&lt;br&gt;C 2 A&lt;br&gt;2 C B</td>
</tr>
<tr>
<td>B</td>
<td>1&lt;br&gt;C 2 A&lt;br&gt;2 A B</td>
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<tr>
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<td>1&lt;br&gt;2 C 3 A&lt;br&gt;3 A B</td>
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<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
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</thead>
<tbody>
<tr>
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<td>A, 2</td>
<td>A, X 3</td>
<td>C</td>
<td>C A</td>
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<tr>
<td>2 C</td>
<td>B, 2</td>
<td>B</td>
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Termination.
Every non-terminated day a job crossed an item off the list.
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Total size of lists?
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Total size of lists? $n$ jobs, $n$ length list.
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Every non-terminated day a job **crossed** an item off the list.
Total size of lists? $n$ jobs, $n$ length list. $n^2$
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Total size of lists? $n$ jobs, $n$ length list. $n^2$
Terminates in $\leq n^2$ steps!
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If on day \( t \) a candidate \( g \) has a job \( b \) on a string, any job, \( b' \), on candidate \( g \)'s string for any day \( t' > t \) is at least as good as \( b \).

Example: Candidate "Alice" has job "Amalgamated Concrete" on string on day 5. She has job "Amalgamated Asphalt" on string on day 7. Does Alice prefer "Almalgamated Asphalt" or "Amalgamated Concrete"?

Improvement Lemma says she prefers "Almalgamated Asphalt".

Day 10: Can Alice have "Amalgamated Asphalt" on her string? Yes. Alice prefers day 10 job as much as day 7 job. Here, \( b = b' \).

Why is lemma true?

Proof Idea: She can always keep the previous job on the string.
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Proof:

$P(0)$ – true. Candidate has $b$ on string.

Assume $P(k)$. Let $b'$ be job on string on day $t + k$.

On day $t + k + 1$, job $b'$ comes back.

Candidate $g$ can choose $b'$, or do better with another job, $b''$. That is, $b' \leq b$ by induction hypothesis.

And $b''$ is better than $b'$ by algorithm.

$\Rightarrow$ Candidate does at least as well as with $b$.

$P(k) = \Rightarrow P(k + 1)$.

And by principle of induction, lemma holds for every day after $t$. 

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Assume $P(k)$. Let $b'$ be job on string on day $t + k$.

On day $t + k + 1$, job $b'$ comes back.

Candidate $g$ can choose $b'$, or do better with another job, $b''$

That is, $b' \leq b$ by induction hypothesis.

And $b''$ is better than $b'$ by algorithm.

$\implies$ Candidate does at least as well as with $b$.

$P(k) \implies P(k + 1)$.

And by principle of induction, lemma holds for every day after $t$.  □
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Contradiction.
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**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.
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\begin{align*}
&b^* \quad \quad \quad \quad \quad \quad \quad g^* \\
&b \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad g
\end{align*}
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\begin{array}{c}
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\downarrow \\
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Assume there is a rogue couple; \((b, g^*)\)

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\(b^*\) prefers \(g^*\) to \(g\).

By improvement lemma, \(g^*\) prefers \(b^*\) to \(b\).

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Good for jobs? candidates?

Is the Job-Proposes better for jobs?
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**Definition:** A pairing is $x$-optimal if $x$’s partner is its best partner in any **stable** pairing.

...and so on for job pessimal, candidate optimal, candidate pessimal.

**Claim:** The optimal partner for a job must be first in its preference list.

True? False! 

Subtlety here: Best partner in any stable pairing. As well as you can be in a globally stable solution!

**Question:** Is there a job or candidate optimal pairing?

Is it possible: $b$-optimal pairing different from the $b'$-optimal pairing! Yes? No?
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Yes? No?
Understanding Optimality: by example.

A: 1,2
B: 1,2

1: A,B
2: B,A

Consider pairing:
(A,1), (B,2).

Stable?
Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.
Also optimal for A, 1 and 2.

A: 1,2
B: 2,1

Pairing S:
(A,1), (B,2).

Stable?
Yes.

Pairing T:
(A,2), (B,1).

Also Stable.

Which is optimal for A?
S

Which is optimal for B?
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Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2  1: B,A
B: 2,1  2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
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A: 1,2 1: B,A
B: 2,1 2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A?
Understanding Optimality: by example.

A:  1,2  
B:  1,2  

Consider pairing: (A, 1), (B, 2).

Stable? Yes.
Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A:  1,2  
B:  2,1  

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Which is optimal for A? S
Understanding Optimality: by example.

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B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
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Optimal for B?
Notice: only one stable pairing.
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A:  1,2  
B:  2,1  

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S  Which is optimal for B? S
Understanding Optimality: by example.

A: 1,2  
B: 1,2  

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2  
B: 2,1  

Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S  
Which is optimal for B? S  
Which is optimal for 1?
Understanding Optimality: by example.

A: 1,2  
  1: A,B
B: 1,2  
  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2  
  1: B,A
B: 2,1  
  2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S  
Which is optimal for B? S
Which is optimal for 1? T
Understanding Optimality: by example.

A: 1,2  1: A,B  
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2  1: B,A  
B: 2,1  2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S  Which is optimal for B? S
Which is optimal for 1? T  Which is optimal for 2?
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
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So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2  1: B,A
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Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S  Which is optimal for B? S
Which is optimal for 1? T  Which is optimal for 2? T
Job Propose and Candidate Reject is optimal!

For jobs?

Theorem: Job Propose and Reject produces a job-optimal pairing.

Proof: Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired. Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ knocks $b$ off of $g$'s string on day $t = \Rightarrow g$ prefers $b^*$ to $b$.

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate. $= \Rightarrow b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$. So $S$ is not a stable pairing.

Notes: $S$ - stable. ($b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle... Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

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$\Rightarrow b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing.

Contradiction.

Notes:
$S$ - stable.
$S^* = (b^*, g) \in S$.
But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...
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Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**

Notes:

*S* - stable.

\((b^*, g^*) \in S\).

But \((b^*, g)\) is a rogue couple!
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For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not:

Let \( t \) be first day job \( b \) gets rejected by its optimal candidate \( g \) who it is paired with in stable pairing \( S \).

\( b^* \) knocks \( b \) off of \( g \)'s string on day \( t \).

By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.

\( \Rightarrow \)

\( b^* \) prefers \( g \) to its partner \( g^* \) in \( S \).

Rogue couple for \( S \).

So \( S \) is not a stable pairing.

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Notes:

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$b^*$ knocks $b$ off of $g$'s string on day $t$.

$b^*$ prefers $g$ to $b$ by choice of $t$.

$b^*$ likes $g$ at least as much as optimal candidate.

$b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.

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Notes:
$S$ - stable.
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\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.
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By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.

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Rogue couple for \( S \).
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By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

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Notes:
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Rogue couple for $S$.

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Used Well-Ordering principle...
**Job Propose and Candidate Reject is optimal!**

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\( b^{*} \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \Rightarrow \) \( g \) prefers \( b^{*} \) to \( b \)

By choice of \( t \), \( b^{*} \) likes \( g \) at least as much as optimal candidate.

\( \Rightarrow \) \( b^{*} \) prefers \( g \) to its partner \( g^{*} \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing. Contradiction.

Notes: \( S \) - stable. \( (b^{*}, g^{*}) \) \( \in \) \( S \). But \( (b^{*}, g) \) is rogue couple!

Used Well-Ordering principle...Induction.
How about for candidates?

Theorem: Job Propose and Reject produces candidate-pessimal pairing.

**T** – pairing produced by JPR.

**S** – worse stable pairing for candidate $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ prefers $b$ to $b^*$.

$T$ is job optimal, so $b$ prefers $g$ to its partner in $S$.

$(g, b)$ is Rogue couple for $S$.

$S$ is not stable.

Contradiction.

Notes: Not really induction.

Structural statement: Job optimality $\Rightarrow$ Candidate pessimality.
How about for candidates?

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$(g,b)$ is Rogue couple for $S$.

$S$ is not stable.

Contradiction.
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

\( T \) – pairing produced by JPR.

\( S \) – worse stable pairing for candidate \( g \).

In \( T \), \((g, b)\) is pair.

In \( S \), \((g, b^\ast)\) is pair.

\( g \) prefers \( b \) to \( b^\ast \).

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How does one make it better for candidates?
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Propose and Reject - stable matching algorithm. One side proposes.
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Jobs Propose $\implies$ job optimal.
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The method was used to match residents to hospitals.
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The method was used to match residents to hospitals. Hospital optimal....

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Another variation: couples.
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Proofs carefully use definition:
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Proofs carefully use definition:
  Optimality proof:
    contradiction of the existence of a better pairing.