Rules of Counting

Zeroth Rule: For any sets $A$ and $B$, if there exists $f:A \rightarrow B$ a bijection, then $|A|=|B|$ (definition).

First Rule: Suppose we are creating an object by $k$ successive choices, where there are $n_i$ options for the $i$th choice. Then there are $n_1 \cdot n_2 \cdots n_k$ possible objects.

Second Rule: Suppose we are creating an object by $k$ choices, the order of which does not matter. If there exists an $m$-to-1 function between $A$, the set of objects created by ordered choices and $B$, the set of objects created by unordered choices, then $|B| = \frac{|A|}{m}$.

Q Let $S = \{1, 2, \ldots, n\}$ and $1 \leq k \leq n$.

How many sequences of $k$ numbers from $S$ are there?

How many sequences of $k$ distinct numbers from $S$ are there?

How many subsets of $k$ distinct numbers from $S$ are there?

How many "bags" of $k$ numbers from $S$ are there?
Combinatorial Proofs

Def. A combinatorial proof counts some carefully chosen set in two different ways to show two expressions are equal.

Thm (Binomial)
\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\]

pp

Thm (Vandermonde's Identity)
\[\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}\]

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**Pf**

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**Pf**
Permutations and Derangements

**Def** A permutation \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \) is

**Ex**

\( Q \) How many permutations of \( S = \{ s_1, \ldots, s_n \} \) are there?

**Def** A derangement is

**Ex**
Derangements (Recursive)

**Theorem:** For \( n > 3 \), the number of derangements of \( \{1, 2, \ldots, n\} \), \( D_n \), satisfies

\[
D_n = (n-1)(D_{n-1} + D_{n-2})
\]

**Proof:** By combinatorial proof.

Deriving this recursion is not easy; nor is solving it. We will consider another approach.
Q. Consider any two sets $A_1$ and $A_2$. What is $|A_1 \cup A_2|$?

\[ A \cup B = A + B - \text{overlap} \]

Q. Consider any three sets $A_1$, $A_2$, $A_3$. What is $|A_1 \cup A_2 \cup A_3|$?

Ex. Out of 50 animals
- 30 can fly
- 12 can swim
- 5 can fly and swim
- 1 can fly and swim

How many animals can do at least one of flying and swimming?
Counting Unions

Q. Consider any two sets $A_1$ and $A_2$. What is $|A_1 \cup A_2|$?

Q. Consider any three sets $A_1$, $A_2$, $A_3$. What is $|A_1 \cup A_2 \cup A_3|$?

Example:
Out of 50 animals
- 30 can fly
- 12 can swim
- 5 can fly and swim
- 5 can do neither

How many animals can do at least one of flying and swimming?
The Principle of Inclusion–Exclusion

Then (Inclusion–Exclusion) Let \( A_1, \ldots, A_n \) be arbitrary finite subsets of some universal set \( A \). Then

\[
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} \left| \bigcap_{i=1}^{n} A_i \right|
\]

Proof. Combinatorially.

Note

\[
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\text{singles}} |A_i| - \sum_{\text{pairs}} |A_i \cap A_j| + \sum_{\text{triples}} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} \left| \bigcap_{i=1}^{n} A_i \right|
\]

In terms

\[
\quad \text{\( \frac{n}{2} \)} \text{ terms} \quad \frac{n}{3} \text{ terms} \quad \frac{n!}{n(k)!} \text{ terms}
\]

The \( k \)-th summation has \( \binom{n}{k} \) terms.

Note. PIE allows us to express a union in terms of intersections.
Derangements (PIE)

Q What is a closed-form expression for $D_n$?