-1 Correction
Regrades open not Monday
More tips on problem solving

Every challenging mathematical problem is its own little world.

Some strategies

1. Set the parameters to extreme values and see what happens.

2. Try out small examples where you can brute force the calculations.

3. Make a guess about what’s going on &
Recap

Previously on CS 70...

1. **Independent events**
   \[ P(ANB) = \]

2. **Intersections of events**
   \[ P(ANBNC) = \]
   
   Easy if \( A, B, C \)

3. **Unions of events**
   \[ P(AUBUC) = P(A) + P(B) + P(C) \]
   
   \[ - P(ANB) - P(ANC) - PCBNC \]
   
   \[ + P(ANBNC) \]

**Union bound:** \( P(AUBUC) \)

When you can't calculate exactly,
Random variables

Common situation: want to measure some quantity of the outcomes in a sample space and answer questions like:

- ........................................
- ........................................
- ........................................

Example: Flip a fair coin 3 times & count the number of heads

<table>
<thead>
<tr>
<th>Possible values</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>HHH</td>
<td></td>
</tr>
<tr>
<td>HHT</td>
<td></td>
</tr>
<tr>
<td>HTH</td>
<td></td>
</tr>
<tr>
<td>THH</td>
<td></td>
</tr>
<tr>
<td>THT</td>
<td></td>
</tr>
<tr>
<td>TTH</td>
<td></td>
</tr>
<tr>
<td>TTT</td>
<td></td>
</tr>
</tbody>
</table>
Suppose \( \Omega \) is a sample space. A random variable \( X \) is

\[
P(X = a) = \frac{w_a}{w_1 + w_2 + w_3 + w_4}
\]

The distribution of \( X \) is that tells us for each possible value of \( X \),

\[
\Omega = \{ w_1, w_2, w_3, w_4 \}
\]
Example: Roll two fair dice and let $X =$

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \\
2 & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \\
3 & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \\
4 & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \\
5 & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \\
6 & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \\
\end{array}
\]

Distribution

$X((1,4)) =$

$X((5,2)) =$
Some types of random variables show up so often that we give them fancy names.

Actually, technically we name...
2.1 Bernoulli Distribution

Have a biased coin which is heads with probability $p$. Flip it once. Define a random variable $X$ by

$$
\text{Distribution of } X
$$

This is called the and we say Fancy name, simple concept

Comment: How to get a simple concept in math named after you?
Example: Roll a fair 6-sided die. Define a random variable $X$ by

**Question**

**Answer**

The point
2.2 Binomial distribution

Have a biased coin which is heads with probability $p$. Flip it $n$ times. Let $X =$

Example $n = 3, \ p = \frac{1}{3}$

Distribution of $X$

This is called the Written
Example: When data is sent over the internet, it is broken up into small chunks called packets.

Each packet is sent separately and usually a few of the packets fail to reach the intended destination.

This is often modelled as

What assumptions are we making?
### Joint distributions

**Def.** Suppose $\Omega$ is a sample space and $X: \Omega \to \mathbb{R}$ and $Y: \Omega \to \mathbb{R}$ are two random variables. The joint distribution of $X$ and $Y$ is

#### Example
Flip a fair coin 3 times. Let

- $X = \#$ of heads
- $Y = \begin{cases} 1 & \text{if 1st flip is H} \\ 0 & \text{if 1st flip is T} \end{cases}$

<table>
<thead>
<tr>
<th>$w$</th>
<th>$X(w)$</th>
<th>$Y(w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HHH</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>HHT</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>HTH</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>HTT</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>THH</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>THT</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>TTH</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>TTT</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Joint distribution:
3.1 Marginal distributions

Suppose we have 2 random variables and we know their joint distribution. How do we...

Example: Suppose we have random variables $X$ and $Y$...

<table>
<thead>
<tr>
<th>Joint distribution of $X$ &amp; $Y$:</th>
<th>$X$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-2$</td>
<td>0</td>
</tr>
<tr>
<td>$X$</td>
<td>0</td>
<td>1/12</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>1/12</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Question

Answer
Suppose we have 2 random variables and we know their joint distribution. How do we find the distribution of $X$ by itself? 

"Marginal distribution for $X$"

For any random variables $X$ and $Y$,

$$P(X = a) =$$

pf
3.2 Independence of random variables

Question: Given distribution of $X$ and distribution of $Y$, how can you compute the joint distribution of $X$ and $Y$?

Answer:

Definition: Random variables $X$ and $Y$ are independent if

Intuitive idea:
Example: Roll 2 fair dice

X = value of 1st roll
Y = value of 2nd roll
Z = sum of the two rolls

Question: X and Y independent?

Answer:

Question: X and Z independent?

Answer:
When you have > 2 random variables

Just like events,

**Def**: Random variables $X_1, X_2, ..., X_n$ are **pairwise independent**

if

**Def**: Random variables $X_1, X_2, ..., X_n$ are **mutually independent**

if

**Common abbreviation**

$X_1, X_2, ..., X_n$ are i.i.d. random variables

**Note**: “Independent” usually means
Combining random variables

Given random variables $X, Y$ it is common to form a new random variable.

Example: Flip a fair coin 3 times

$X = \begin{cases} 1 & \text{HHT} \\ 2 & \text{HTH} \\ 3 & \text{TTH} \\ 0 & \text{THT} \\ 0 & \text{TTT} \end{cases}$

$Y = \begin{cases} 1 & \text{HHT} \\ 1 & \text{HTH} \\ 0 & \text{TTH} \\ 1 & \text{THT} \\ 0 & \text{TTT} \end{cases}$

Define $Z = \begin{cases} w(X(w)) + y(y(w)) & \text{Question} \\ \text{Answer} \end{cases}$

<table>
<thead>
<tr>
<th>$w$</th>
<th>$X(w)$</th>
<th>$Y(w)$</th>
<th>$Z(w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HHH</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>HHT</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>HTH</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>THH</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>HTT</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>THT</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>TTH</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>TTT</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
4. Expected value

Question: I flip a fair coin 3 times and pay you $1 for each H. How much would you pay me to play this game?

Answer

Def: If $X$ is a random variable, the expected value of $X$, written $E(X)$, is defined by

$$E(X) = \sum \text{probability} \times \text{value}$$

Note:

If you do something with expected value $a$ and you repeat it many times (independently) then in the long run the average of those times will be close to $a$. 
If $X$ is a random variable, the expected value of $X$, written $E(X)$, is defined by

$$E(X) = \sum_{a \in \text{range}(X)} a \cdot P(X=a)$$

Example: Flip a coin 3 times. $X =$

$$E(X) = \sum_{\omega \in S^3} X(\omega) \cdot P(\omega)$$

Prop
Example \( X \sim \text{Bin}(n, p) \)

\[ E(X) = \] 

Enter, the hero:
⑤ Linearity of Expectation

Then for any random variables $X_1, X_2,..., X_n$

$$E(X_1 + X_2 + ... + X_n) =$$
Expected value of binomial distribution, revisited

Have a coin which is heads with probability $p$

Flip it $n$ times

$X = \# \text{ of heads}$