Announcements

Regrade requests

If you lost points because you selected the wrong pages for a problem, you should submit a regrade request.

Midsemester survey

Please fill it out if you can.
Goal for today

Two more common distributions

Derive their basic properties

Why?

Want to build probabilistic models of situations in science or technology
1 Geometric distribution

Have a biased coin which is heads with probability $p$
Flip it until you get heads

$X = \text{Example}$
Have a biased coin which is heads with probability $p$.

Flip it until you get heads.

$x = \# \text{ of flips}$

**Example**

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$x$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>TTTTH</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>TTTTTTTH</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**Distribution of $x$**

Possible values:

$P(x=i) =$

This distribution is called the ________ and we write ________.
Reminder: $X \sim \text{Geometric}(p) \Rightarrow P(X = i) = (1-p)^{i-1}p$

**Question:** What is $\sum_{i=1}^{\infty} P(X = i)$?

**Answer:**

$$\sum_{i=1}^{\infty} P(X = i) =$$

**Lesson**
1.1 Expected value of geometric distribution

Thm IF \( X \sim \text{Geometric}(p) \) then \( E(x) = \frac{1}{p} \).

\[
p^k \quad E(x) =
\]

\[
E(x) =
\]

Warning: Can only manipulate infinite series like this when they are
If $X \sim \text{Geometric}(p)$ then $E(X) = \frac{1}{p}$.

**Example:** I roll a fair 6-sided die until I get a 6. If it takes $n$ rolls,

**Question:** How much are you willing to pay to play this game with me?

**Answer**
Example: Coupon Collector’s Problem

\[ n \text{ types of Pokémon} \]
Each day you catch a random Pokémon

Define \[ X = \]
Define $X = \#$ of days until you have one of each type

What is $E(X)$?

Define new random variables

Note:

Example

Day 1  Day 2  Day 3  Day 4  Day 5  Day 6
$X = \# \text{ of days until we get all } n \text{ types}$

For $i = 1, 2, \ldots, n$, $X_i = \# \text{ of days after we have } i-1 \text{ types until we have } i \text{ types}$

Note: $X = X_1 + X_2 + \ldots + X_n$

$E(X) =$

**Upshot:**

**What is $E(X_i)$?**

**Distribution of $X_i$:**

$\Rightarrow E(X_i) =$
### 1.3 Variance of geometric distribution

**Theorem:** If $X \sim \text{Geometric}(p)$ then $\text{Var}(X) = \frac{1-p}{p^2}$

**Proof:** \[ \text{Var}(X) = \]

\[ E(X^2) = \]

\[ \Rightarrow \]
Theorem: If $X \sim \text{Geometric}(p)$ then $\text{Var}(X) = \frac{1-p}{p^2}$.

Proof (continued): $\text{Var}(X) = E(X^2) - E(X)^2 = \frac{1-p}{p^2}$.

Have: $E(X^2) - (1-p)E(X)^2 = \sum_{i=1}^{\infty} (2i-1) (1-p)^{i-1} p$

$= \Rightarrow$

Comment:
② Poisson distribution

\[ \text{Poisson} \neq \text{Poison} \]

**Def:** A random variable \( X \) has the **Poisson distribution** with parameter \( \lambda \) if the distribution of \( X \) is

we write \( X \sim \text{Poisson}(\lambda) \) to indicate this

**Comment:** Bernoulli

Binomial

Geometric

Poisson
2.1 Intuitive meaning of Poisson distribution

Poisson distribution: \( P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \) for \( k = 0, 1, 2, \ldots \)

Intuitive meaning
Some event occurs

\# of times it occurs in any 2 disjoint intervals are

\( X = \)

What random variables have Poisson distribution?

①

②
A cool example

Sometimes, cosmic rays
When they do, they can cause

This is a real problem!

How many bits do we expect to flip this way per year?
Poisson distribution: \( P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \) for \( k = 0, 1, 2, \ldots \)

**Uses of the Poisson distribution**

**Use #1:** Model various naturally occurring situations in science, technology, etc.

**Use #2:** Approximation to the binomial distribution

One formalization of this:

\[ \text{Thm} \]
2.2 Calculating things about the Poisson distribution

Suppose $X \sim \text{Poisson}(\lambda)$

What are $E(X)$ and $\text{Var}(X)$?

Warm-up $\sum_{k=0}^{\infty} \Pr(X = k) = \frac{e^{-\lambda}}{k!}$

Comment

Lesson:
Thm: If $X \sim \text{Poisson}(\lambda)$ then $E(X) = \lambda$

Consider $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
Thm  If $X \sim \text{Poisson}(\lambda)$ then $\text{Var}(X) = \lambda$

Comment: Common way to check if Poisson distribution is a good model for a situation:
What does $\text{Poisson}(\lambda)$ look like?

As $\lambda \to \infty$, later we will see this is explained by the following fact:

If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent, then
Then if $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent then $X + Y \sim \text{Poisson}(\lambda + \mu)$

\textbf{pf} \quad \text{Let } k \in \mathbb{N}, \\
\text{Need to show}
Then for $n = 1, 2, 3, \ldots$ let $X_n \sim \text{Bin}(n, \frac{1}{n})$.

Then for any $k \in \mathbb{N}$

$$\lim_{n \to \infty} P(X_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

**Intuition** Poisson ($\lambda$)

- expected number of events:

1

2

3
Then for $n = 1, 2, 3, \ldots$ let $X_n \sim \text{Bin}(n, \frac{1}{n})$.

Then for any $k \in \mathbb{N}$

$$\lim_{n \to \infty} P(X_n = k) = \frac{\Delta^k}{k!} e^{-\lambda}$$

Proof
<table>
<thead>
<tr>
<th>Distribution</th>
<th>Expression</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli</td>
<td>$X \sim Bernoulli(p)$</td>
<td>$E(X) = p$</td>
<td>$Var(X) = p(1-p)$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$X \sim Bin(n, p)$</td>
<td>$E(X) = np$</td>
<td>$Var(X) = np(1-p)$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$X \sim Geometric(p)$</td>
<td>$E(X) = \frac{1}{p}$</td>
<td>$Var(X) = \frac{L-2}{p^2}$</td>
</tr>
<tr>
<td>Poisson</td>
<td>$X \sim Poisson(\lambda)$</td>
<td>$E(X) = \lambda$</td>
<td>$Var(X) = \lambda$</td>
</tr>
</tbody>
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