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Planar Five Color theorem.

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Types of graphs.

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- Complete Graphs.
- Trees.
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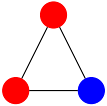

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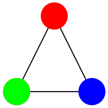

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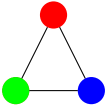

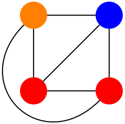

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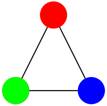

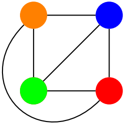

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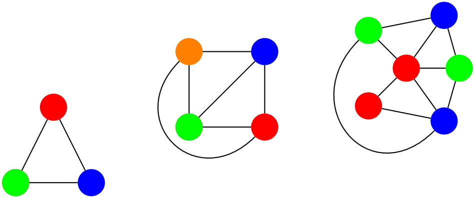

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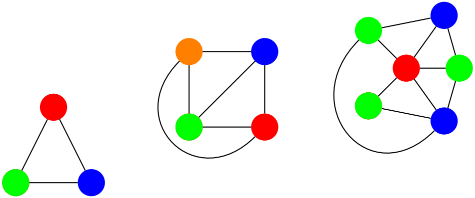

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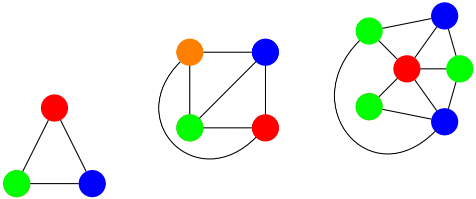

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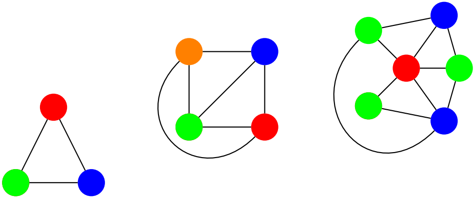

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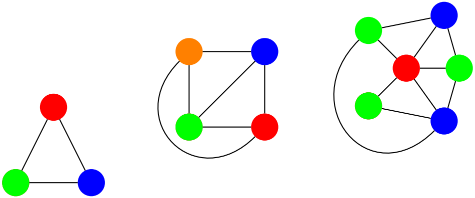

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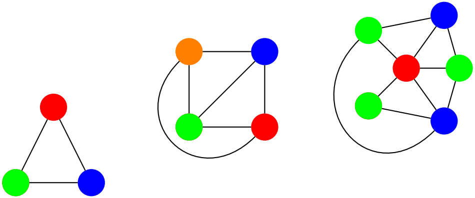

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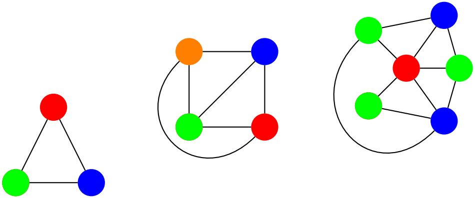

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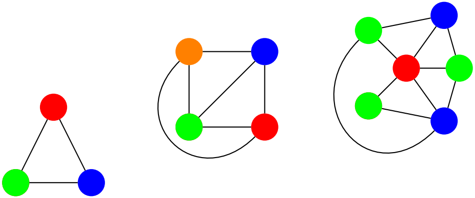

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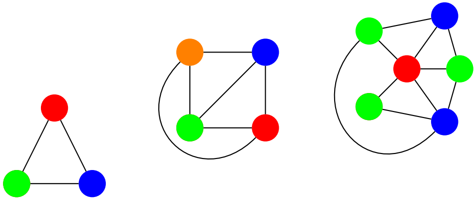

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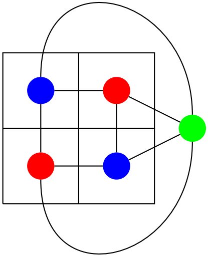
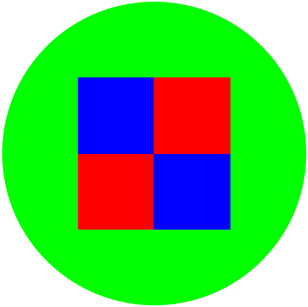
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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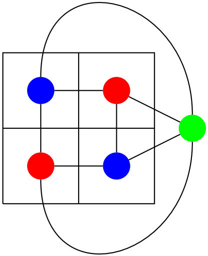

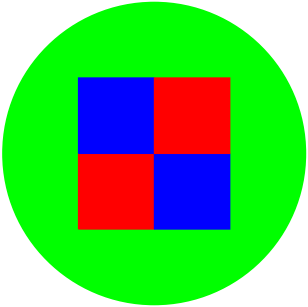

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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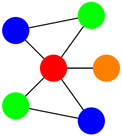
Five color theorem: prelimnary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.



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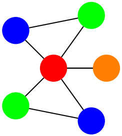

Look at only green and blue.

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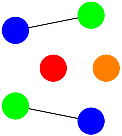

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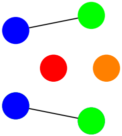

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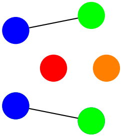

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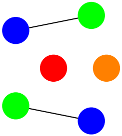

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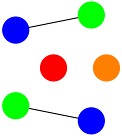

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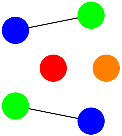

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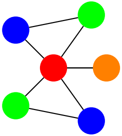

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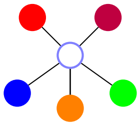

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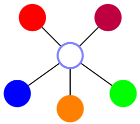

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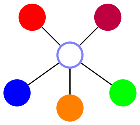

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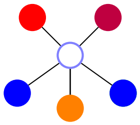

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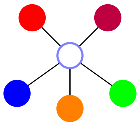

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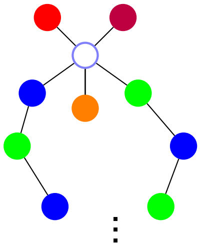

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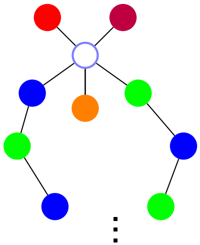

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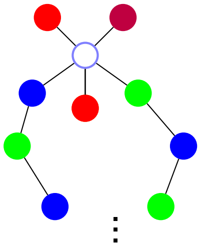

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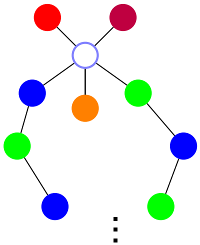

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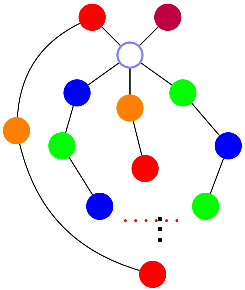

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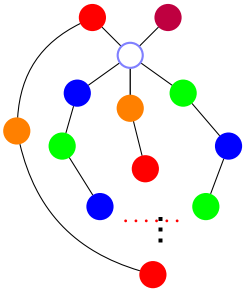

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Planar.

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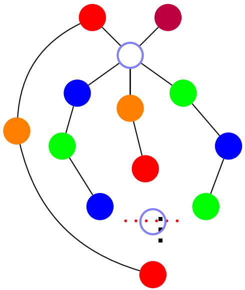
Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



Assume neighbors are colored all differently.
Otherwise done.
Switch green to blue in component.
Done. Unless blue-green path to blue.
Switch orange to red in its component.
Done. Unless red-orange path to red.

Planar. $\Longrightarrow$ paths intersect at a vertex!

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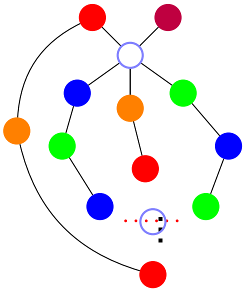

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What color is it?

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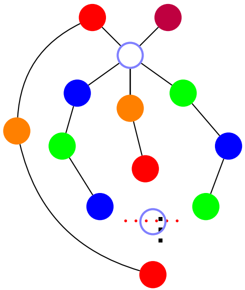

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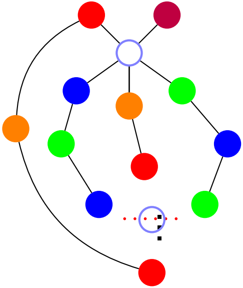

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What color is it?
Must be blue or green to be on that path.

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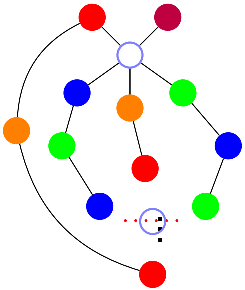

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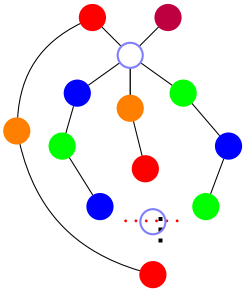

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What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction.

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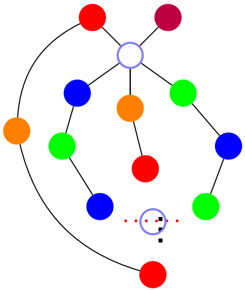

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What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
And recolor "center" vertex.

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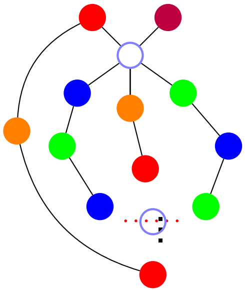

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And recolor "center" vertex. $\square$

Four Color Theorem

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof:

Four Color Theorem

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Proof: Not Today!

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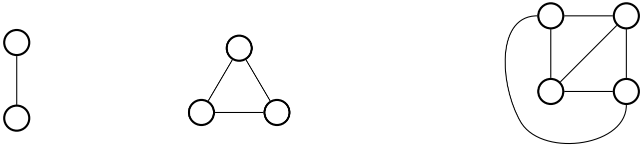
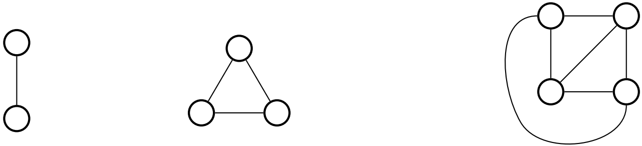
Complete Graph.



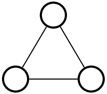
K_n complete graph on n vertices.

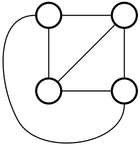
Complete Graph.



K_n complete graph on n vertices.
All edges are present.

Complete Graph.



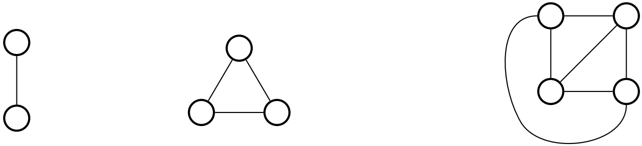


K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Complete Graph.



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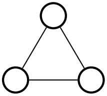
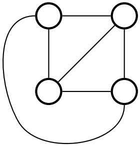
Each vertex is adjacent to every other vertex.

Complete Graph.





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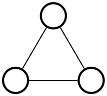
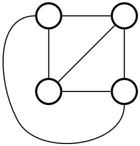
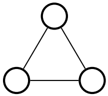


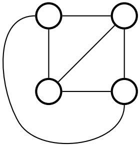


K_n complete graph on n vertices.
  All edges are present.
  Everyone is my neighbor.
  Each vertex is adjacent to every other vertex.

How many edges?

Complete Graph.





K_n complete graph on n vertices.

All edges are present.

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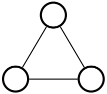
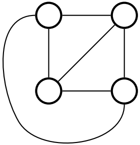
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n - 1$ edges.

Complete Graph.





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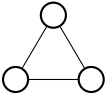
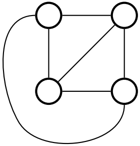
Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1)$.

Complete Graph.

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$\Longrightarrow$ Number of edges is $n(n-1)/2$.

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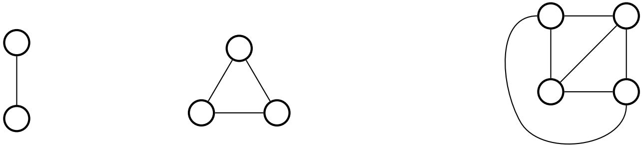

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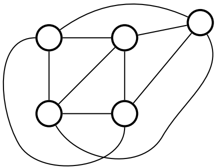
$\implies$ Number of edges is $n(n-1)/2$.

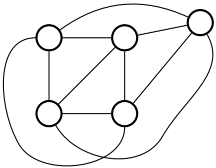
Remember sum of degree is $2|E|$.

K_4 and K_5



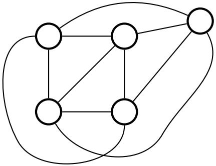
K_5 is not planar.

K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!

K_4 and K_5

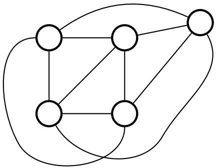


K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it!

K_4 and K_5



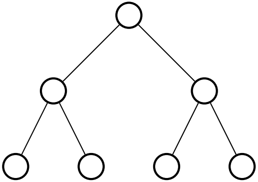
K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it! We did!

A Tree, a tree.

Graph $G = (V, E)$.
Binary Tree!



More generally.

Trees.

Definitions:

Trees.

Definitions:

A connected graph without a cycle.

Trees.

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A connected graph without a cycle.
A connected graph with $|V|-1$ edges.

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A connected graph without a cycle.
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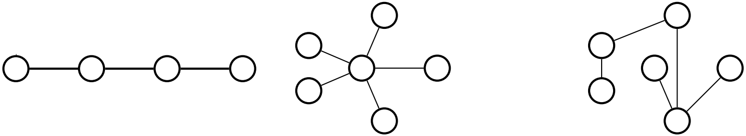

no cycle and connected?

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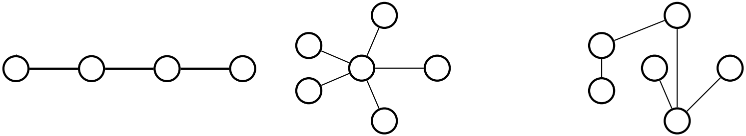

no cycle and connected? Yes.

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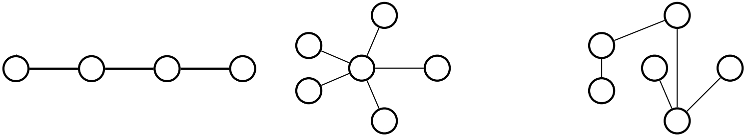

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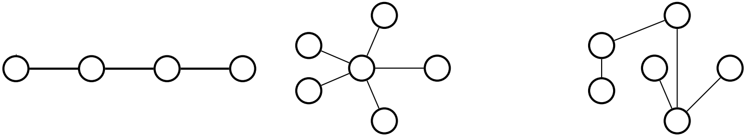

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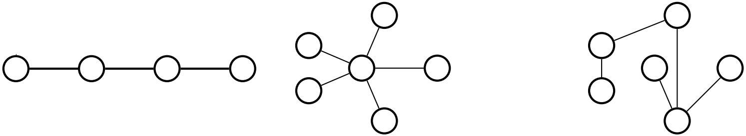

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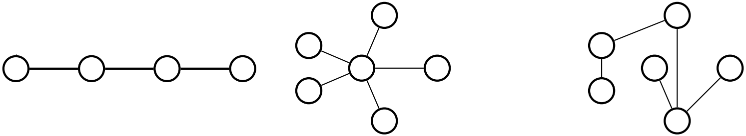

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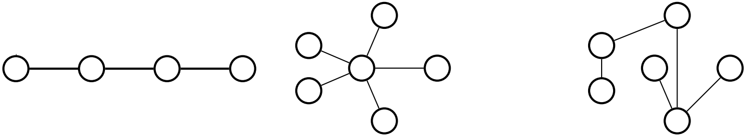

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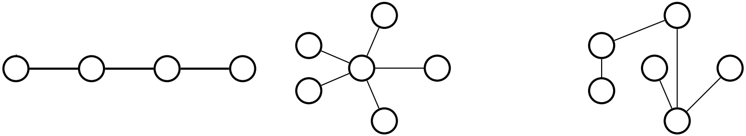

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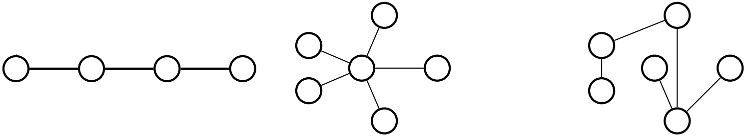

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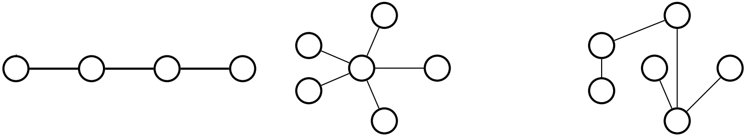

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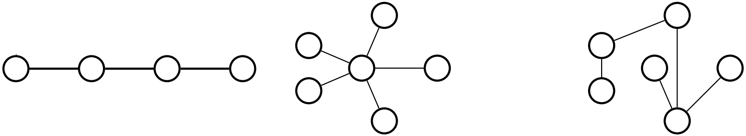

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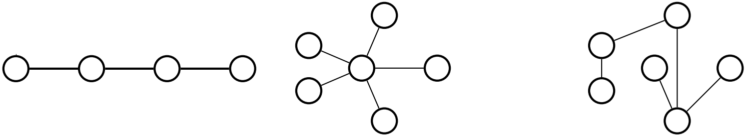

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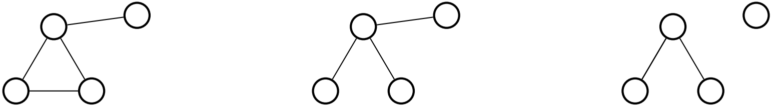
To tree or not to tree!



Equivalence of Definitions.

Theorem:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."

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Lemma: If v is a degree 1 in connected graph G, $G - v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,

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Proof:
For $x \neq v, y \neq v \in V$,
there is path between x and y in G since connected.

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For $x \neq v, y \neq v \in V$,
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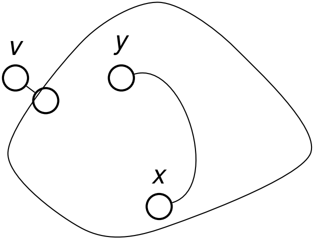

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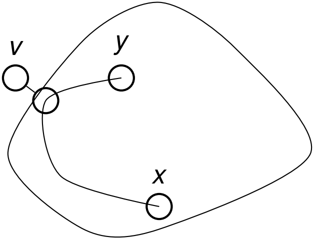

Proof of only if.

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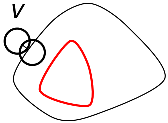

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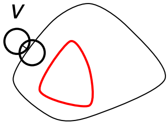
Proof of $\implies$: By induction on $|V|$.



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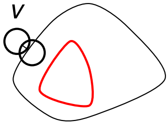

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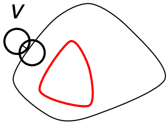
Base Case: $|V|=1$. $0=|V|-1$ edges and has no cycles.

Proof of only if.



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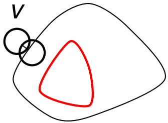

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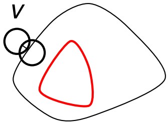

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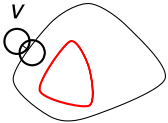
Proof of $\implies$: By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Induction Step:
Claim: There is a degree 1 node.

Proof of only if.



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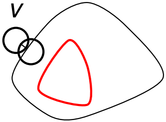

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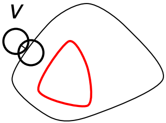
Sum of degrees is $2|V|-2$

Proof of only if.



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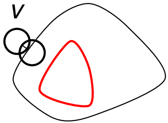
Proof of only if.



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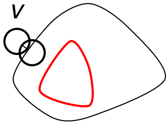
Not everyone is bigger than average!

Proof of only if.



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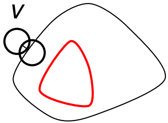

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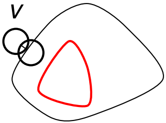

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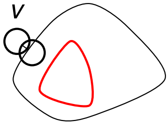

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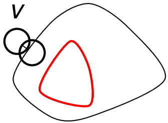

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And no cycle in G since degree 1 cannot participate in cycle.

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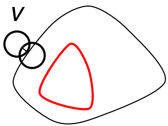

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Walk from a vertex using untraversed edges.

Proof of if

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Proof of Claim:
Can't visit more than once since no cycle.

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Proof of Claim:
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Entered. Didn’t leave.

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Removing node doesn't create cycle.

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New graph is connected.

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By induction $G - v$ has $|V| - 2$ edges.
G has one more or $|V| - 1$ edges.

Proof of if

Thm:
"G is connected and has no cycles"
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Walk from a vertex using untraversed edges.
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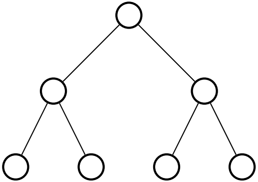
Tree's fall apart.

Thm: There is one vertex whose removal disconnects $|V|/2$ nodes from each other.



Idea of proof.

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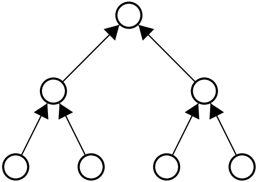

Idea of proof.
Point edge toward bigger side.

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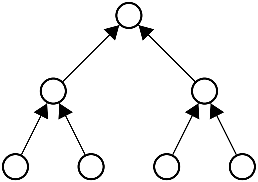

Idea of proof.
Point edge toward bigger side.
Remove center node.

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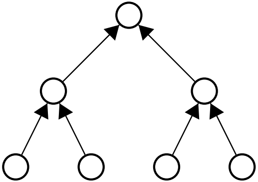

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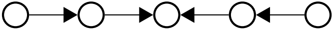

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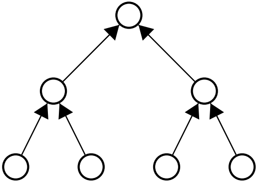

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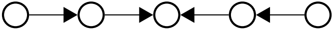

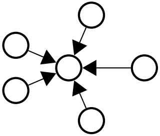

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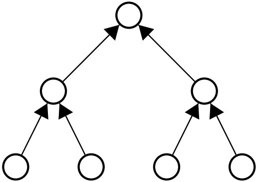

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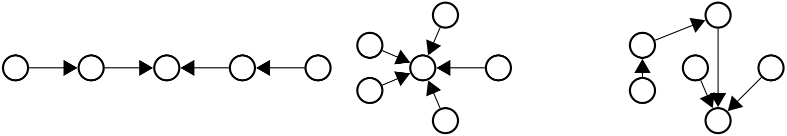

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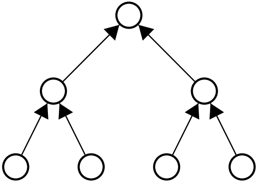

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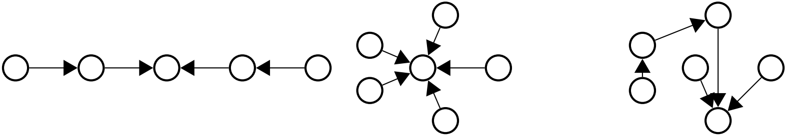

Hypercubes.

Complete graphs, really connected!

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

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Trees,

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$|V|(|V|-1)/2$

Trees, few edges. $(|V|-1)$

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Hypercubes. Really connected. $|V|\log|V|$ edges!

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Also represents bit-strings nicely.

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$G=(V,E)$

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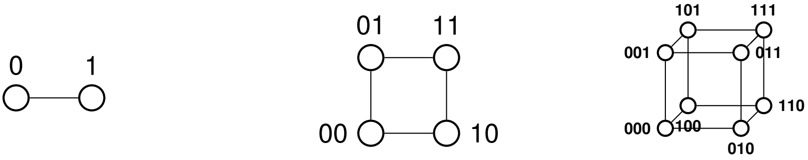

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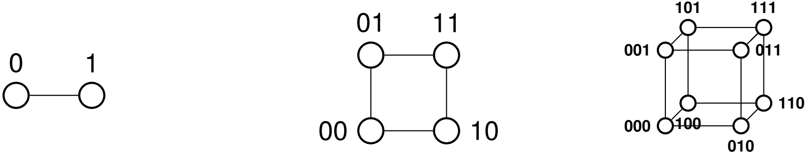

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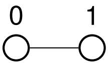

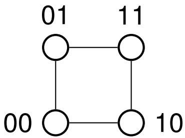

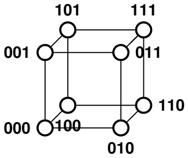

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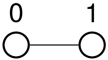

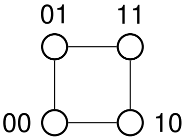

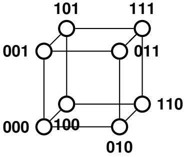

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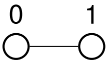

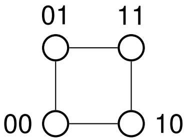

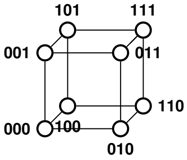

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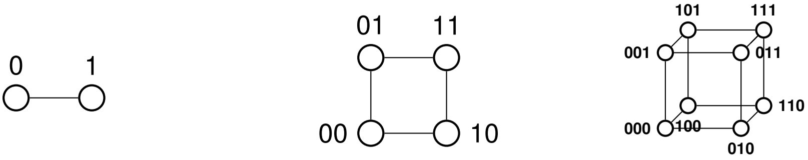

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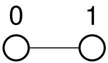

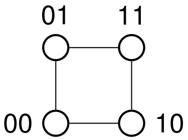

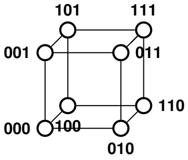

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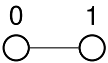

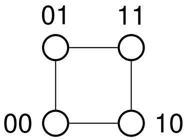

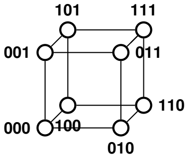

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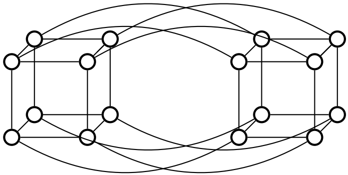

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

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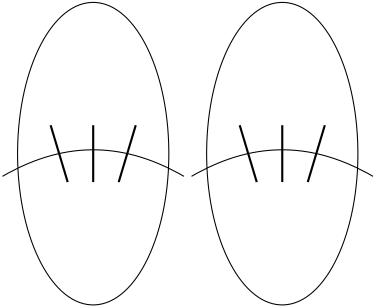

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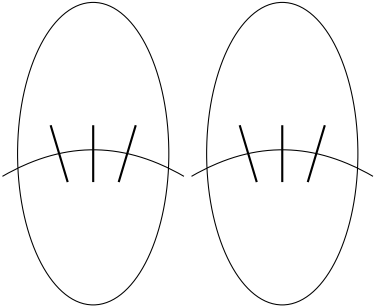

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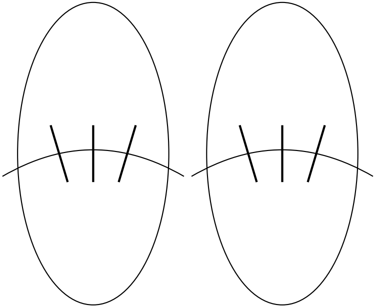

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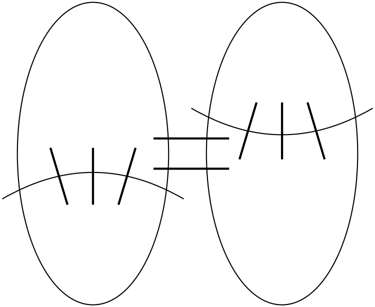

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Induction Step. Case 2.

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Proof: Induction Step. Case 2.

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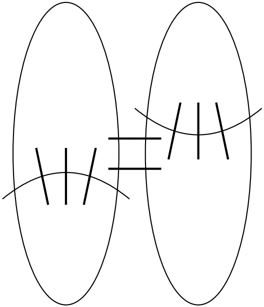

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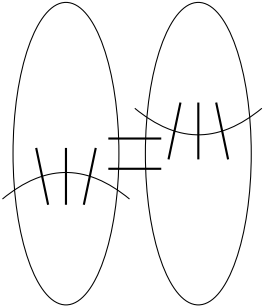

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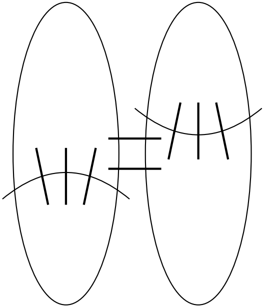

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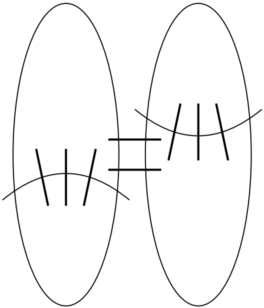

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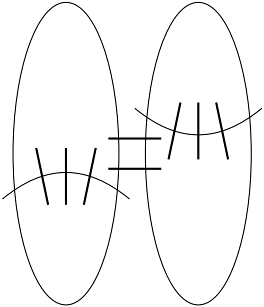

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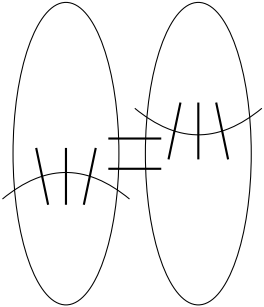

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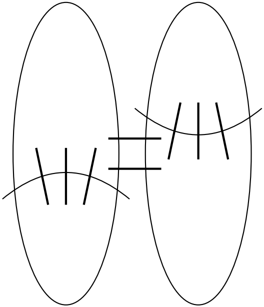

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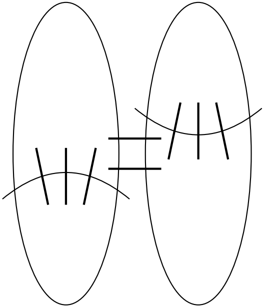

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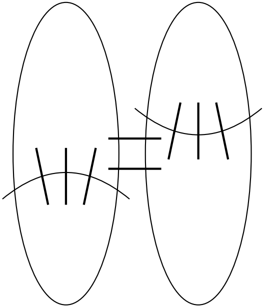

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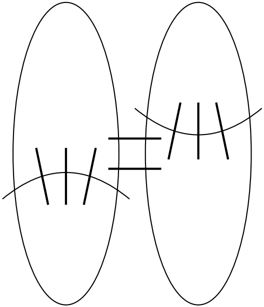

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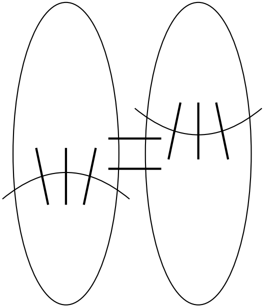

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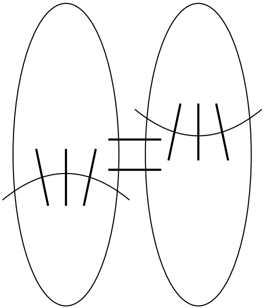

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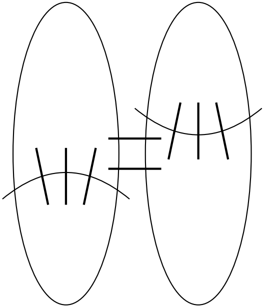

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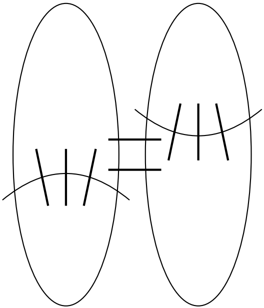

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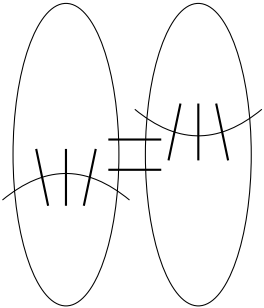

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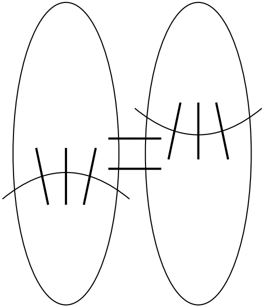

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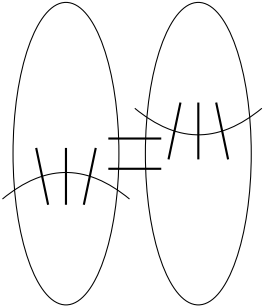

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Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

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Have a nice weekend!