1. Intro to Continuous Probability
2. Probability Density Function (PDF)
3. Continuous Uniform R.V. (running example)
4. Analogs Between Continuous & Discrete
5. Cumulative Distribution Function (CDF)
6. Conditioning on an Event
7. Mixed Random Variables
8. Conditional Expectation
9. Two Envelope Paradox

Announcements
- Midsemester Survey Feedback.

Homework is hard and takes lots of time but really helps you learn!
Continuous Probability

Random Variable $X$:

function $X: \Omega \rightarrow \mathbb{R}$

$X = k$ corresponds to $\{ \omega \in \Omega : X(\omega) = k \}$

Coin flip $\rightarrow \{ H, T \} \quad |\Omega| = 2$

Poisson RV $\rightarrow \{ 0, 1, 2, \ldots \} \quad |\Omega| = |\mathbb{N}|$

In the real world, we are often more interested in situations where $\Omega$ is uncountably infinite in size.
Example Situation

Circumference : 2

\[ X \text{ is a continuous R.V.} \]
\[ X \in [0, 2) \]
\[ P(X = 0) = 0 \]

If we assign \( P(X = 0) \) a positive probability, then summing over all \( k \in (0, 2) \) gives us something bigger than 1.

\[ P(X \text{ lands on } \frac{\pi}{2}) = \frac{1}{4} \]

1) For a continuous random variable,

\[ P(X = \omega) = 0 \quad \text{for all } \omega \in \Omega \]

2) Intervals do have probability.
A probability density function (pdf) for a real-valued r.v. $X$ is a function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

1. $\forall x \in \mathbb{R}, \ f(x) \geq 0$ (nonnegative)

2. $\int_{-\infty}^{\infty} f(x) \, dx = 1$

and:

3. $P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx$

Note:

$P(X = a) = P(a \leq X \leq a) = \int_{a}^{a} f(x) \, dx = 0$

This implies that

$P(a \leq X \leq b) = P(a < x < b)$

(endpoints don't contribute probability)
Probability Density?

\[ f_x(x) \]

\[ t \quad t + \delta \]

\[ x \]

Observe: If \( \delta > 0 \) is very small, then \( f_x(x) \) is roughly constant on \([t, t + \delta]\).

\[ P(t \leq X \leq t + \delta) = \int_t^{t + \delta} f_x(x) \, dx \]

\[ \approx f_x(t) \cdot \delta \]

\[ f_x(t) = \frac{P(t \leq X \leq t + \delta)}{\delta} \]

"probability density"
Example: $X \sim \text{Unif}(0, 2)$

What is the pdf of $X$?

We know by def. of pdf:

\[
\int_{-\infty}^{\infty} f_X(x) \, dx = 1
\]

\[
\int_{0}^{2} c \, dx = 1
\]

\[
2 \cdot c = 1
\]

\[
c = \frac{1}{2}
\]

\[
f_X(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{1}{2} & \text{if } x \in [0, 2] \\
0 & \text{if } x > 2
\end{cases}
\]
Example: \( X \sim \text{Unif}(a, b) \)

\[
f_X(x) = \begin{cases} 
0 & \text{if } x \in (-\infty, a) \\
\frac{1}{b-a} & \forall x \in [a, b] \\
0 & \forall x \in (b, \infty)
\end{cases}
\]

What does uniformly random mean?

It means that probability is proportional to length.
Analogies

Summations $\rightarrow$ Integrals
PMF $\rightarrow$ PDF

\[ E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \, dx \]

\[ \text{Var}(X) = E[X^2] - (E[X])^2 \]

\[ = \int_{-\infty}^{\infty} x^2 f(x) \, dx - \left( \int_{-\infty}^{\infty} x f(x) \, dx \right)^2 \]

LOTOS (continuous)

PMF for Poisson r.v.

\[ P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \]

Probability mass function
Example \( X \sim \text{Unif}(0, 2) \)

\[
\begin{align*}
E[X] &= \int_{-\infty}^{\infty} x \cdot f(x) \, dx \\
&= \int_{0}^{2} x \cdot \frac{1}{2} \, dx = \frac{x^2}{2} \cdot \frac{1}{2} \Big|_{0}^{2} = \frac{4}{4} - 0 = 1
\end{align*}
\]

\[
E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_{0}^{2} x^2 \cdot \frac{1}{2} \, dx = \frac{8}{6}
\]

\[
\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{8}{6} - 1^2 = \frac{1}{3}
\]
The CDF is a bridge between discrete and continuous cases.

The cumulative distribution function (CDF) of a random variable $X$ is the function $F$ where:

$$F(x) = \mathbb{P}(X \leq x)$$

CDF is always a probability.

In the continuous case,

$$F(x) = \int_{-\infty}^{x} f(x) \, dx$$
Example: \( X \sim \text{Unif}(0,2) \)

What is the CDF of \( X \)?

\[
F(x) = P(X \leq x)
\]

\[
= \int_{-\infty}^{x} f_X(x) \, dx
\]

\[
= \int_{0}^{x} \frac{1}{2} \, dx = \frac{1}{2} \times \int_{0}^{x} = \frac{x}{2}
\]

(if \( x \in [0,2] \))

\[
P(X \leq x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{x}{2} & \text{if } x \in [0,2] \\
1 & \text{if } x > 2
\end{cases}
\]
Cumulative Distribution Function

The CDF has the following key properties:

1. \( \lim_{x \to -\infty} F(x) = 0 \)
2. \( \lim_{x \to +\infty} F(x) = 1 \)
3. \( F(x) \) is monotonically increasing
4. \( F(x) \) uniquely characterizes the distribution of \( X \)

\[ P(a \leq X \leq b) = \int_a^b f(x) \, dx = F(b) - F(a) \]  
(by FTC)
Example: $X \sim \text{Unif}(0, 2)$

Recall: CDF of $X$

$$F(x) = P(X \leq x) = \begin{cases} 
0 & \forall x < 0 \\
\frac{x}{2} & \forall x \in [0, 2] \\
1 & \forall x > 2 
\end{cases}$$

$$\lim_{x \to -\infty} F(x) = 0 \quad \checkmark$$

$$\lim_{x \to +\infty} F(x) = 1 \quad \checkmark$$

Monotonically increasing \checkmark

Break: 4:06 PM
Recovering PDF/PMF from CDF

Take a derivative.

Continuous Case:
\[ f(x) = \frac{dF(x)}{dx} \quad (\text{by FTC}) \]

Discrete Case:
\[ P(x) = \frac{F(x) - F(x-1)}{x - (x-1)} \]
Example: $X \sim \text{Unif}(0, 2)$

CDF to PDF:

$$F(X \leq x) = \begin{cases} 
0 & x < 0 \\
\frac{x}{2} & x \in [0, 2] \\
1 & x > 2 
\end{cases}$$

$f(x) = \begin{cases} 
0 & x < 0 \\
\frac{1}{2} & x \in [0, 2] \\
0 & x > 2 
\end{cases}$
Discrete vs Continuous Recap:

<table>
<thead>
<tr>
<th>Discrete</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMF $P(x=x)$ (&quot;mass&quot;)</td>
<td>PDF $f_x(x)$ (&quot;density&quot;)</td>
</tr>
<tr>
<td>CDF $P(x \leq x)$</td>
<td>CDF $P(x \leq x)$</td>
</tr>
<tr>
<td>$E[X] = \sum_x x \cdot P(X=x)$</td>
<td>$E[X] = \int_{-\infty}^{\infty} x \cdot f_x(x) , dx$</td>
</tr>
</tbody>
</table>

Note:

The PDF of continuous random variables can be greater than 1.

Consider $X \sim \text{Unif} [0, \frac{1}{2}]$.
Conditioning on an Event

\[ X \text{ continuous R.V.} \]

\[ A \text{ set of values } X(w) \]

\[ A \text{ event } P(A) > 0 \]

A is the set of outcomes such that \( X(w) \) is in fancy \( A \)

\[ P(X \in A) = P(A) \]

Note: This has been corrected. By defining the fancy \( A \) first as a set of values the random variable could take on, and then \( A \) as all the outcomes that result in a value in the set fancy \( A \), there are no outcomes outside of \( A \) that result in a value in fancy \( A \), so the two probabilities are equal.

\[ f_{X|A}(x) \cdot \delta = P(x \leq X \leq x + \delta | X \in A) \]

\[ = \frac{P(x \leq X \leq x + \delta \cap X \in A)}{P(X \in A)} \]

\[ \Rightarrow f_{X|A} = \left\{ \begin{array}{ll} \frac{f_X(x)}{P(A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{array} \right. \]

Discrete Case:

If \( X \) is a discrete R.V.

\[ P_{X|A}(x) = \left\{ \begin{array}{ll} \frac{P(X=x)}{P(A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{array} \right. \]
Example

\[ X \sim \text{Unif}[0, 2] \]

Let \( A \) be the event \( X \in [0, 1] \)

\[
f_{X|A}(x) = \begin{cases} 
\frac{f_X(x)}{P(A)} & \text{if } x \in [0, 1] \\
0 & \text{o.w.}
\end{cases}
\]

\[
= \begin{cases} 
\frac{\frac{1}{2}}{\frac{1}{2}} & \text{if } x \in [0, 1] \\
0 & \text{o.w.}
\end{cases}
\]

\[
= \begin{cases} 
1 & \text{if } x \in [0, 1] \\
0 & \text{o.w.}
\end{cases}
\]
Mixed Random Variables

Some random variables are neither discrete or continuous, but rather a combination of the two.

Example:

You flip a fair coin. If it is heads, you get a reward of 0.5 points. If it is tails, you spin a wheel to get a point value in $[0, 2]$. Let $X$ be the mixed random variable representing the amount of points you have at the end of the game.
Conditional Expectation

Let $A$ be an event, and $X$ a continuous random variable.

$$
E[X|A] =
$$

This also holds in the discrete case, just use pmf instead of pdf.

It follows:

$$
E[X] = E[X|A] \cdot P(A) + E[X|A^c] \cdot P(A^c)
$$

Proof left as exercise.

Hint: (1) $E[X] = \sum_x x \cdot P(X=x)$

(2) Use law of total probability on $P(X=x)$
Conditional Expectation Example.

Coin Flip

\[ \begin{align*}
\text{H} & \quad 0.5 \quad \text{p(H)} \\
\text{T} & \quad 0.5 \\
\end{align*} \]

What is \( E[X] \)?

\[ E[X] = \]

\[ E[X|H] = \]

\[ E[X] = E[X|H] \cdot P(H) + E[X|H'] \cdot P(H') \]

\[ = \]

\[ = \]
Two Envelope Paradox:

Identical in appearance, weight, etc.
One envelope has $x$, the other has $2x$.
$x$ is a positive real number.
You are given an unopened envelope at random.
You then get a chance to keep it (and the money inside) or switch to the other one.

What should you do?
Argument 1: It doesn't matter (by symmetry)

Argument 2:
Let $A$ be the amount in the envelope you are given. Let $B$ be the amount in the other one.

$E[B] = $