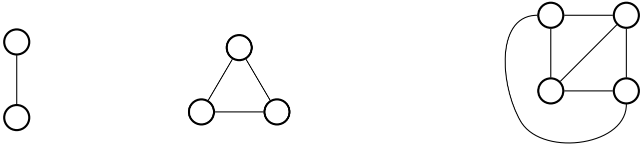
Today.

Types of graphs.

- Complete Graphs.
- Trees.
- Planar Graphs.

Complete Graph.



K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

How many edges?

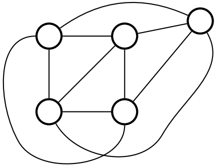
Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1)$.

$\implies$ Number of edges is $n(n-1)/2$.

Remember sum of degree is $2|E|$.

K_5



K_5 is not planar.

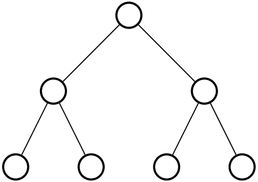
Cannot be drawn in the plane without an edge crossing!

Prove it! We will!

A Tree, a tree.

Graph $G = (V, E)$.

Binary Tree!



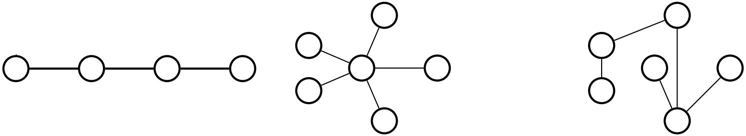
More generally.

Trees.

Definitions:

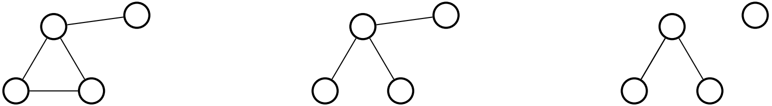
A connected graph without a cycle.
A connected graph with $|V|-1$ edges.
A connected graph where any edge removal disconnects it.
A connected graph where any edge addition creates a cycle.

Some trees.



no cycle and connected? Yes.
$|V|-1$ edges and connected? Yes.
removing any edge disconnects it. Harder to check. but yes.
Adding any edge creates cycle. Harder to check. but yes.

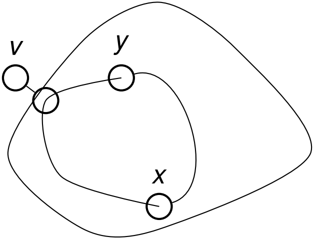
To tree or not to tree!



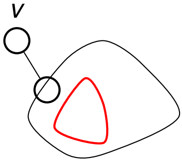
Equivalence of Definitions.

Theorem:
"G connected and has $|V| - 1$ edges" $\equiv$
"G is connected and has no cycles."

Lemma: If v is degree 1 in connected G, then $G - v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,
there is path between x and y in G since connected.
and does not use v (degree 1)
$\implies G - v$ is connected. $\square$



Proof of only if.



Thm:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."

Proof of $\implies$: By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

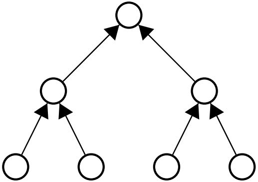
Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|V| - 2$
Average degree $2 - 2/|V|$
Not everyone is bigger than average! $\square$
By degree 1 removal lemma, $G - v$ is connected.
$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction
$\implies$ no cycle in $G - v$.
And no cycle in G since degree 1 cannot participate in cycle. $\square$

Proof of if

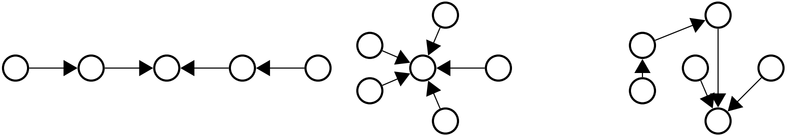
Thm:
"G is connected and has no cycles"
$\implies$ "G connected and has $|V| - 1$ edges"
Proof:
Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Degree 1 vertex.
Proof of Claim:
Can't visit more than once since no cycle.
Entered. Didn't leave. Only one incident edge. □
Removing node doesn't create cycle.
New graph is connected.
Removing degree 1 node doesn't disconnect from Degree 1 lemma.
By induction $G - v$ has $|V| - 2$ edges.
G has one more or $|V| - 1$ edges. □

Tree's fall apart.

Thm: There is one vertex whose removal disconnects $|V|/2$ nodes from each other.

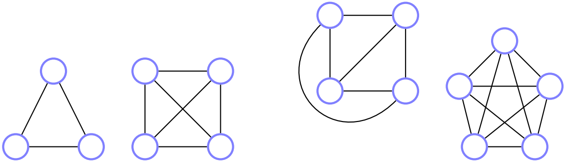


Idea of proof.
Point edge toward bigger side.
Remove center node.

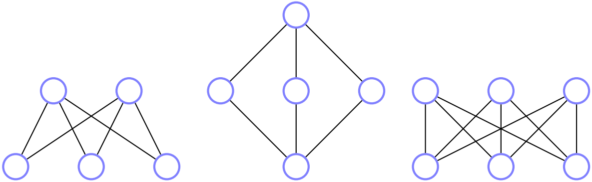


Planar graphs.

A graph that can be drawn in the plane without edge crossings.

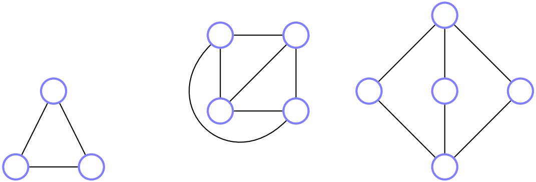


Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or K_5? No! Why? Later.



Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$. No. Why? Later.

Euler's Formula.



Faces: connected regions of the plane.

How many faces for
triangle? 2
complete on four vertices or K_4? 4
bipartite, complete two/three or $K_{2,3}$? 3

v is number of vertices, e is number of edges, f is number of faces.

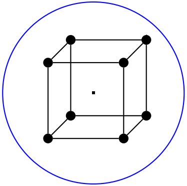
Euler's Formula: Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2$!
K_4: $4 + 4 = 6 + 2$!
$K_{2,3}$: $5 + 3 = 6 + 2$!

Examples = 3! Proven! Not!!!!

Euler and Polyhedron.

Greeks knew formula for polyhedron.



Faces? 6. Edges? 12. Vertices? 8.

Euler: Connected planar graph: $v + f = e + 2$.
$8 + 6 = 12 + 2$.

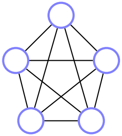
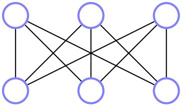
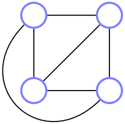
Greeks couldn't prove it. Induction? Remove vertice for polyhedron?

Polyhedron without holes $\equiv$ Planar graphs.

Surround by sphere.
Project from point inside polytope onto sphere.
Sphere $\equiv$ Plane! Topologically.

Euler proved formula thousands of years later!

Euler and planarity of K_5 and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

We consider graphs where $v \geq 3$.

Each face is adjacent to edge at least 3 times for simple graph.

$\geq 3f$ face-edge adjacencies.

Each edge is adjacent to (at most) two faces.

$\leq 2e$ face-edge adjacencies.

$\implies 3f \leq 2e$ *for any planar graph with more than* 2 *vertices*

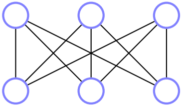
... or $\frac{2}{3}e \geq f$.

+ Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

K_5 Edges? $4 + 3 + 2 + 1 = 10$. Vertices? 5.

$10 \not\leq 3(5) - 6 = 9$. $\implies$ K_5 is not planar.

$K_{3,3}$ non-planarity.



Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!

Planar? No.

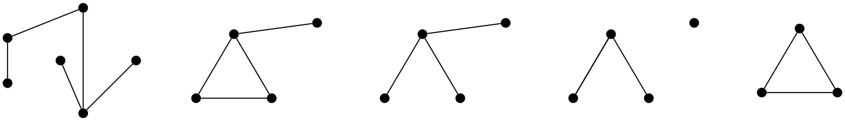
No cycles that are triangles.
  Cycles of length ≥ 4.
At least $4f$ face-edge adjacencies,
  and at most $2e$.
.... $4f \leq 2e$ for any bipartite planar graph.
Euler: $v + \frac{1}{2}e \geq e + 2 \implies e \leq 2v - 4$ for bipartite planar graph

$9 \not\leq 2(6) - 4$. $\implies$ $K_{3,3}$ is not planar!

Tree.

A tree is a connected acyclic graph.

To tree or not to tree!



Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.
Vertices/Edges. Recall: $e = v - 1$ for tree.

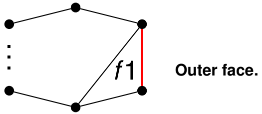
One face for trees!

Euler works for trees: $v + f = e + 2$.
$v + 1 = v - 1 + 2$

Euler's formula.

Euler: Connected planar graph has $v+f=e+2$.

Proof sketch: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:
  If it is a tree. Done.
  If not a tree.
    Find a cycle. Remove edge.



Joins two faces.
New graph: v-vertices. $e-1$ edges. $f-1$ faces. Planar.
$v+(f-1)=(e-1)+2$ by induction hypothesis.
Therefore $v+f=e+2$. □

Summary

Graphs, trees, complete graphs, planar graphs.

Euler’s formula.

Have a nice weekend!