Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.
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1. Isoperimetric inequality for hypercube.
2. Modular Arithmetic.
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1. Isoperimetric inequality for hypercube.

2. Modular Arithmetic.
   Clock Math!!!
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1. Isoperimetric inequality for hypercube.

2. Modular Arithmetic.
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3. Inverses for Modular Arithmetic: Greatest Common Divisor.
Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.
2. Modular Arithmetic.  
   Clock Math!!!
3. Inverses for Modular Arithmetic: Greatest Common Divisor. Division!!!
Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.

2. Modular Arithmetic.
   Clock Math!!!

3. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!

4. Euclid’s GCD Algorithm.
Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.

2. Modular Arithmetic.
   Clock Math!!!

3. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!

4. Euclid’s GCD Algorithm.
   A little tricky here!
Isoperimetry.

For 3-space:
Isoperimetry.

For 3-space:

The sphere minimizes surface area to volume.
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The sphere minimizes surface area to volume.

Surface Area: $4\pi r^2$, Volume: $\frac{4}{3} \pi r^3$. 
Isoperimetry.

For 3-space:

The sphere minimizes surface area to volume.

Surface Area: $4\pi r^2$, Volume: $\frac{4}{3}\pi r^3$.

Ratio: $1/3r = \Theta(V^{-1/3})$. 
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Graphical Analog: Cut into two pieces and find ratio of edges/vertices on small side.
For 3-space:

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Surface Area: $4\pi r^2$, Volume: $\frac{4}{3}\pi r^3$.

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Graphical Analog: Cut into two pieces and find ratio of edges/vertices on small side.

Tree: $\Theta(1/|V|)$. 
Isoperimetry.

For 3-space:

The sphere minimizes surface area to volume.

Surface Area: \(4\pi r^2\), Volume: \(\frac{4}{3}\pi r^3\).

Ratio: \(1/3r = \Theta(V^{-1/3})\).

Graphical Analog: Cut into two pieces and find ratio of edges/vertices on small side.

Tree: \(\Theta(1/|V|)\).

Hypercube: \(\Theta(1)\).
For 3-space:

The sphere minimizes surface area to volume.

Surface Area: $4\pi r^2$, Volume: $\frac{4}{3}\pi r^3$.

Ratio: $1/3r = \Theta(V^{-1/3})$.

Graphical Analog: Cut into two pieces and find ratio of edges/vertices on small side.

Tree: $\Theta(1/|V|)$.

Hypercube: $\Theta(1)$.

Surface Area is roughly at least the volume!
Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.
Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An \( n \)-dimensional hypercube consists of a 0-subcube (1-subcube) which is a \( n - 1 \)-dimensional hypercube with nodes labelled \( 0x \) \((1x)\) with the additional edges \((0x, 1x)\).
A 0-dimensional hypercube is a node labelled with the empty string of bits.

An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$. 
Hypercube: Can’t cut me!

**Theorem:**
Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$.

**Terminology:**
- $(S, V - S)$ is a cut.
- $(E \cap S \times (V - S))$ are cut edges.

**Restatement:** For any cut in the hypercube, the number of cut edges is at least the size of the small side.
Hypercube: Can’t cut me!

**Thm:** Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$;
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Terminology:
(S, V – S) is cut.
**Thm:** Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

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Terminology:
(S, V − S) is cut.
(E ∩ S × (V − S)) - cut edges.
Thm: Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

Terminology:
- $(S, V - S)$ is cut.
- $(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
Cuts in graphs.

$S$ is red, $V - S$ is blue.
Cuts in graphs.

S is red, $V - S$ is blue.

What is size of cut?
Cuts in graphs.

S is red, $V - S$ is blue.

What is size of cut?

Number of edges between red and blue.
Cuts in graphs.

$S$ is red, $V - S$ is blue.

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Number of edges between red and blue. 4.
Cuts in graphs.

$S$ is red, $V - S$ is blue.

What is size of cut?

Number of edges between red and blue. 4.

Hypercube: any cut that cuts off $x$ nodes has $\geq x$ edges.
Proof of Large Cuts.

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Base Case: \(n = 1\)
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

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Base Case: \(n = 1\) \(V = \{0,1\}\).
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Base Case: \(n = 1\) \(V = \{0, 1\}\).
- \(S = \{0\}\) has one edge leaving.
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Proof:
Base Case: \(n = 1\) \(V = \{0,1\}\).
\(S = \{0\}\) has one edge leaving. \(|S| = \phi\) has 0.
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Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.
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Use recursive definition into two subcubes.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

- Two cubes connected by edges.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.
**Induction Step Idea**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

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Case 1: Count edges inside subcube inductively.

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**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).
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**Proof: Induction Step.**
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

Proof: Induction Step.
Recursive definition:

\[
\begin{align*}
H_0 &= (V_0, E_0), \\
H_1 &= (V_1, E_1), \\
H &= (V_0 \cup V_1, E_0 \cup E_1 \cup E_x), \\
S &= S_0 \cup S_1
\end{align*}
\]

Case 1: \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\).
Both \(S_0\) and \(S_1\) are small sides.
So by induction.
Edges cut in \(H_0\) \(\geq |S_0|\).
Edges cut in \(H_1\) \(\geq |S_1|\).
Total cut edges \(\geq |S_0| + |S_1| = |S|\).
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step.

Recursive definition:

\[ H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.} \]
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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S = S_0 \cup S_1 \text{ where } S_0 \text{ in first, and } S_1 \text{ in other.}
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Induction Step

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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)

Both \(S_0\) and \(S_1\) are small sides. So by induction.

Edges cut in \(H_0 \geq |S_0|\).
**Induction Step**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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Both \(S_0\) and \(S_1\) are small sides. So by induction.

- Edges cut in \(H_0 \geq |S_0|\).
- Edges cut in \(H_1 \geq |S_1|\).
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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Both \(S_0\) and \(S_1\) are small sides. So by induction.

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Total cut edges \(\geq |S_0| + |S_1| = |S|\).
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

Proof: Induction Step.
 Recursive definition:
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H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.}
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 Edges cut in \(H_0 \geq |S_0|\).
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Total cut edges \(\geq |S_0| + |S_1| = |S|\). \(\square\)
**Induction Step. Case 2.**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

- \(|S_0| \geq |V_0|/2.

  **Recall Case 1:** \(|S_0|, |S_1| \leq |V|/2

  \(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2\).

Edges in \(E_x\) connect corresponding nodes.

Total edges cut:

\[\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0| - |V_0|/2 \geq |S|\]

Also, case 3 where \(|S_1| \geq |V|/2\) is symmetric.
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\[ |S_0| \geq \frac{|V_0|}{2}. \]

Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2} \]
\[ |S_1| \leq \frac{|V_1|}{2} \]
\[ \text{since } |S| \leq \frac{|V|}{2}. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]

\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]

\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]

Edges in \(E_x\) connect corresponding nodes.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]

\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]

\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]

\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq \frac{|V_0|}{2}. \]

Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2} \]
\[ |S_1| \leq \frac{|V_1|}{2} \text{ since } |S| \leq \frac{|V|}{2}. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2} \]
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Edges in \(E_x\) connect corresponding nodes.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: 

\[|S_0|, |S_1| \leq |V|/2\]

\[|S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2.\]

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\]

\[\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Edges in \(E_x\) connect corresponding nodes.

\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


\(|S_0| \geq |V_0|/2.\)

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2.\)

\(\implies \geq |S_1|\) edges cut in \(E_1.\)

\(|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\)

\(\implies \geq |V_0| - |S_0|\) edges cut in \(E_0.\)

Edges in \(E_x\) connect corresponding nodes.

\(\implies = |S_0| - |S_1|\) edges cut in \(E_x.\)
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[
|S_0| \geq \frac{|V_0|}{2}.
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Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2}\)

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|S_1| \leq \frac{|V_1|}{2} \text{ since } |S| \leq \frac{|V|}{2}.
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\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

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|S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2}
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\[\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Edges in \(E_x\) connect corresponding nodes.

\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]

Total edges cut:
**Induction Step. Case 2.**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

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|S_0| \geq \frac{|V_0|}{2}.
\]

Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2}

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|S_1| \leq \frac{|V_1|}{2} \text{ since } |S| \leq \frac{|V|}{2}.
\]

\[
\implies \geq |S_1| \text{ edges cut in } E_1.
\]

\[
|S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2}
\]

\[
\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.
\]

Edges in \(E_x\) connect corresponding nodes.

\[
\implies = |S_0| - |S_1| \text{ edges cut in } E_x.
\]

Total edges cut:

\[
\geq
\]
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

| \(|S_0| \geq |V_0|/2.\) |
| \(|S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2.\) |
| \(\implies \geq |S_1| \text{ edges cut in } E_1.\) |
| \(|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\) |
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Edges in \(E_x\) connect corresponding nodes.

\(\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\)

Total edges cut:
\(\geq |S_1|\)
**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof:** Induction Step. Case 2.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$S_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0 - S$</td>
<td>$V_1 - S$</td>
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</table>

$|S_0| \geq \frac{|V_0|}{2}$.

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$|S_1| \leq \frac{|V_1|}{2}$ since $|S| \leq \frac{|V|}{2}$.

$\implies \geq |S_1|$ edges cut in $E_1$.

$|S_0| \geq \frac{|V_0|}{2} \implies |V_0 - S| \leq \frac{|V_0|}{2}$

$\implies \geq |V_0| - |S_0|$ edges cut in $E_0$.

Edges in $E_x$ connect corresponding nodes.

$\implies = |S_0| - |S_1|$ edges cut in $E_x$.

Total edges cut:

$\geq |S_1| + |V_0| - |S_0|$
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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Total edges cut:

\(\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\)

\(|V_0| = |V|/2 \geq |S|.\)

Also, case 3 where \(|S_1| \geq |V|/2\) is symmetric.
Hypercube has large cuts proof uses these ideas:
(A) If cuts are same size on two sides it works by induction.
(B) Uses the fact that it is planar.
(C) Recursive definition of hypercube.
(D) If different size, can count edges between to subcubes.
(E) Applies Euler’s formula.
Hypercube proof: poll

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(A),(D), and (E).
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).
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Yes/No Computer Programs \(\equiv\) Boolean function on \( \{0, 1\}^n \)
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).

Central area of study in computer science!

Yes/No Computer Programs \( \equiv \) Boolean function on \( \{0, 1\}^n \)

Central object of study.
Modular Arithmetic.

Applications: cryptography, error correction.
Key ideas for modular arithmetic.

Theorem: If \( d | x \) and \( d | y \), then \( d | (y - x) \).
Key ideas for modular arithmetic.

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Proof:
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$x = ad$, $y = bd$, 

Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y - x)$.

Proof:
\[ x = ad, \quad y = bd, \]
\[ (x - y) = (ad - bd) = d(a - b) \implies d|(x - y). \]
Key ideas for modular arithmetic.

Theorem: If \( d \mid x \) and \( d \mid y \), then \( d \mid (y - x) \).

Proof:
\[
x = ad, \quad y = bd,
\]
\[
(x - y) = (ad - bd) = d(a - b) \implies d\mid(x - y).
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Theorem: Every number \( n \geq 2 \) can be represented as a product of primes.
Key ideas for modular arithmetic.

Theorem: If $d| x$ and $d| y$, then $d| (y - x)$.

Proof:

$x = ad$, $y = bd$,

$(x - y) = (ad - bd) = d(a - b) \implies d| (x - y)$. 

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction. 

(Uniqueness? Later.)
What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler’s formula.
(C) The definition of a prime number.
(D) Euclid’s Lemma.
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(A) and (C)
Modular Arithmetic.
Clock Math

If it is 1:00 now.
Clock Math

If it is 1:00 now.
What time is it in 2 hours?
If it is 1:00 now.
What time is it in 2 hours? 3:00!
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours?
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours?
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!

Actually 4:00.
16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to addition/subtraction of 12.

What time is it in 100 hours?
101:00!
or 5:00.
101 = 12 \times 8 + 5.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.
Custom is only to use the representative in \{12, 1, ..., 11\} (Almost remainder, except for 12 and 0 are equivalent.)
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
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What time is it in 100 hours? 101:00! or 5:00.
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Day of the week.

This is Thursday is February 11, 2021.
Day of the week.

This is Thursday is February 11, 2021. What day is it a year from then?
Day of the week.

This is Thursday is February 11, 2021.

What day is it a year from then? on February 11, 2022?
Day of the week.

This is Thursday is February 11, 2021. What day is it a year from then? on February 11, 2022? Number days.

...
This is Thursday is February 11, 2021.

What day is it a year from then? on February 11, 2022?

Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
This is Thursday is February 11, 2021.
What day is it a year from then? on February 11, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then.
day 9 or day 2 or Tuesday.
25 days from then.
day 29 or day 1.
29 = (7) 4 + 1
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then
is day 1 which is Monday!
What day is it a year from then?
Next year is not a leap year.
So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7
leaves quotient of 52 and remainder 5.
369 = 7(52) + 5
or February 11, 2022 is a Friday.
Day of the week.

This is Thursday, February 11, 2021.
What day is it a year from then, on February 11, 2022?
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Day 4+365 or day 369.
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subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 5.

or February 11, 2022 is a Friday.
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Years and years...

80 years?
Years and years...

80 years? 20 leap years.
Years and years...

80 years? 20 leap years. $366 \times 20$ days

Today is day 4. It is day 4 $+ 366 \times 20 + 365 \times 60$.

Equivalent to?

Hmm. What is remainder of 366 when dividing by 7?

$52 \times 7 + 2$.

What is remainder of 365 when dividing by 7?

1.

Today is day 4.

Get Day: 4 $+ 2 \times 20 + 1 \times 60$.

Remainder when dividing by 7?

$104 = 14 \times 7 + 6$.

Or February 11, 2101 is Saturday!

Further Simplify Calculation:

20 has remainder 6 when divided by 7.

60 has remainder 4 when divided by 7.

Get Day: 2 $+ 2 \times 6 + 1 \times 4$.

Or Day 6.

February 11, 2101 is Saturday.

"Reduce" at any time in calculation!

$18 / 34$
Years and years...

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Years and years...

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Hmm.
Years and years...

80 years? 20 leap years. 366 × 20 days
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It is day 4 + 366 × 20 + 365 × 60. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7?

20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.

Today is day 4 + 2 × 6 + 1 × 4 = 18.
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Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$. 

Mod 7 equivalence classes:

\{... , -7, 0, 7, 14, ...\}

\{... , -6, 1, 8, 15, ...\}

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.
Modular Arithmetic: refresher.

\(x\) is congruent to \(y\) modulo \(m\) or “\(x \equiv y \pmod{m}\)’’ if and only if \((x - y)\) is divisible by \(m\).
...or \(x\) and \(y\) have the same remainder w.r.t. \(m\).
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...or $x = y + km$ for some integer $k$.  

Mod 7 equivalence classes:
\{...,-7,0,7,14,...\} 
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... 

Useful Fact:
Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.

or “$a \equiv c \pmod{m}$ and $b \equiv d \pmod{m} = \Rightarrow a + b \equiv c + d \pmod{m}$ and $a \cdot b \equiv c \cdot d \pmod{m}$”

Proof:
If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.
If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.
Therefore, $a + b = c + d + (k + j)m$ since $k + j$ is integer.
$= \Rightarrow a + b \equiv c + d \pmod{m}$.

Can calculate with representative in $\{0,...,m-1\}$. 


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\[ x \text{ is congruent to } y \text{ modulo } m \text{ or } \langle x \equiv y \pmod{m} \rangle \]
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Mod 7 equivalence classes:
{\ldots, -7, 0, 7, 14, \ldots} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots
Modular Arithmetic: refresher.

*x* is congruent to *y* modulo *m* or “*x* ≡ *y* (mod *m*)” if and only if (*x* − *y*) is divisible by *m*.

...or *x* and *y* have the same remainder w.r.t. *m*.

...or *x* = *y* + *km* for some integer *k*.

Mod 7 equivalence classes:

{…, −7, 0, 7, 14, …}  {…, −6, 1, 8, 15, …} ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent *x* and *y*.
Modular Arithmetic: refresher.

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\[\implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m}\]"
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\[ \Rightarrow a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m} \]”

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\{\ldots, -7, 0, 7, 14, \ldots\}  \{\ldots, -6, 1, 8, 15, \ldots\}  \ldots

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\[ \Rightarrow a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} \)”

Proof: If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Modular Arithmetic: refresher.

- **$x$ is congruent to $y$ modulo $m$** or “$x \equiv y \pmod{m}$”
  - if and only if $(x - y)$ is divisible by $m$.
  - ...or $x$ and $y$ have the same remainder w.r.t. $m$.
  - ...or $x = y + km$ for some integer $k$.

Mod 7 equivalence classes:

{...,$−7,0,7,14,...$} {...,$−6,1,8,15,...$} ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.

or “ $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$

$\implies a + b \equiv c + d \pmod{m}$ and $a \cdot b \equiv c \cdot d \pmod{m}$”

**Proof:** If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.
If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.

Therefore,
Modular Arithmetic: refresher.

\[ x \text{ is congruent to } y \text{ modulo } m \text{ or } “x \equiv y \pmod{m}” \]
if and only if \((x - y)\) is divisible by \(m\).
...or \(x\) and \(y\) have the same remainder w.r.t. \(m\).
...or \(x = y + km\) for some integer \(k\).

Mod 7 equivalence classes:
\[
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots
\]

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \(x\) and \(y\).

or “\(a \equiv c \pmod{m}\) and \(b \equiv d \pmod{m}\)
\[\implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m}\]”

**Proof:** If \(a \equiv c \pmod{m}\), then \(a = c + km\) for some integer \(k\).
If \(b \equiv d \pmod{m}\), then \(b = d + jm\) for some integer \(j\).
Therefore, \(a + b = c + d + (k + j)m\)
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$”
if and only if $(x - y)$ is divisible by $m$.
...or $x$ and $y$ have the same remainder w.r.t. $m$.
...or $x = y + km$ for some integer $k$.

Mod 7 equivalence classes:
\{...,-7,0,7,14,...\} \{...,-6,1,8,15,...\} ...

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.

or “$a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$
\[ \implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m} \]”

Proof: If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.
If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.
Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
Modular Arithmetic: refresher.

\( x \text{ is congruent to } y \text{ modulo } m \) or “\( x \equiv y \pmod{m} \)”
if and only if \( (x - y) \) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)

\[ \implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m} \]

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore, \( a + b = c + d + (k + j)m \) and since \( k + j \) is integer.
\[ \implies a + b \equiv c + d \pmod{m}. \]
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or \( "x \equiv y \pmod{m}" \)
if and only if \( (x - y) \) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\( \{..., -7, 0, 7, 14, ...\} \quad \{..., -6, 1, 8, 15, ...\} \quad ... \)

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or \( " a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\( \implies a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} \)

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore, \( a + b = c + d + (k + j)m \quad \text{and since } k + j \text{ is integer.} \)
\( \implies a + b \equiv c + d \pmod{m} \).
Modular Arithmetic: refresher.

**x is congruent to y modulo m** or “\(x \equiv y \pmod{m}\)” if and only if \((x - y)\) is divisible by \(m\).

...or \(x\) and \(y\) have the same remainder w.r.t. \(m\).
...or \(x = y + km\) for some integer \(k\).

Mod 7 equivalence classes:
\(\{\ldots, -7, 0, 7, 14, \ldots\}\)  \(\{\ldots, -6, 1, 8, 15, \ldots\}\) ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \(x\) and \(y\).

or “\(a \equiv c \pmod{m}\) and \(b \equiv d \pmod{m}\)
\[\implies a + b \equiv c + d \pmod{m}\) and \(a \cdot b \equiv c \cdot d \pmod{m}\)”

**Proof:** If \(a \equiv c \pmod{m}\), then \(a = c + km\) for some integer \(k\).
If \(b \equiv d \pmod{m}\), then \(b = d + jm\) for some integer \(j\).
Therefore, \(a + b = c + d + (k + j)m\) and since \(k + j\) is integer.
\[\implies a + b \equiv c + d \pmod{m}\).

Can calculate with representative in \(\{0, \ldots, m - 1\}\).
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x, m) \]
Notation

\[ x \pmod{m} \text{ or } \text{mod}(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
Notation

\[ x \text{ (mod } m) \text{ or } \text{mod } (x, m)\]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
\[ - \text{ remainder of } x \text{ divided by } m \text{ in } \{0, \ldots, m-1\}. \]

\[ \mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod(29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 \]
Notation

\[ x \pmod{m} \text{ or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \mod (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 \]
Notation

$x \ (\text{mod} \ m)$ or $\mod \ (x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

$$\mod \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m$$

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

$$\mod \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4$$
Notation

\[ x \pmod{m} \] or \[ \text{mod} (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = x = 5 \]
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} \ (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor\right) \times 12 = 29 - (2) \times 12 = 5 \]

Work in this system.
**Notation**

$x \pmod{m}$ or $\text{mod } (x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m - 1\}$.

$$\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m$$

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

$$\text{mod } (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \mathbf{5}$$

Work in this system.

$a \equiv b \pmod{m}$. 

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Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4
\]

Work in this system.

\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).
Notation

\[ x \text{ (mod } m\text{) or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \mod (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod (29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = \boxed{5} \]

Work in this system.

\[ a \equiv b \text{ (mod } m\text{).} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]

\[
\text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4 = 5
\]

Work in this system.

\[ a \equiv b \pmod{m} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

6 \equiv
Notation

\( x \pmod{m} \) or \( \text{mod} \, (x, m) \)

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} \, (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \, (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 = 5
\]

Work in this system.

\( a \equiv b \pmod{m} \).

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \)
Notation

\[ x \pmod{m} \text{ or } \operatorname{mod}(x, m) \]

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\operatorname{mod}(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\[
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]

\[
\operatorname{mod}(29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4
\]

Work in this system.

\( a \equiv b \pmod{m} \).

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \)
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = \mathbf{5} \]

Work in this system.
\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

Modulus is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \).
Notation

\[ x \equiv b \pmod{m} \] or \[ \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \{0, \ldots, m - 1\}.

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 = 5 \]

Work in this system.

\( a \equiv b \pmod{m} \).

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]

\[
\text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \text{X} = 5
\]

Work in this system.
\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)
\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
\[ 6 = 3 + 3 \]
Notation

\(x \pmod{m}\) or \(\text{mod } (x, m)\)
- remainder of \(x\) divided by \(m\) in \(\{0, \ldots, m-1\}\).

\[\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m\]

\(\left\lfloor \frac{x}{m} \right\rfloor\) is quotient.

\[\text{mod } (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 \equiv 5\]

Work in this system.

\(a \equiv b \pmod{m}\).
Says two integers \(a\) and \(b\) are equivalent modulo \(m\).

**Modulus** is \(m\)

\(6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}\).

\(6 = 3 + 3 = 3 + 10\)
**Notation**

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[
\mod(29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = \mathbf{5}
\]

Work in this system.

\[ a \equiv b \pmod{m}. \]

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]

\[ 6 = 3 + 3 = 3 + 10 \pmod{7}. \]
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4 \]

Work in this system.

\[ a \equiv b \pmod{m} \]

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \]

\[ 6 = 3 + 3 = 3 + 10 \pmod{7} \]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x,m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x,m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[
\text{mod} (29,12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \underline{X} = 5
\]

Work in this system.
\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
\[ 6 = 3 + 3 = 3 + 10 \pmod{7}. \]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
But probably won’t take off points,
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod(29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 \]

Work in this system.

\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
\[ 6 = 3 + 3 = 3 + 10 \pmod{7}. \]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
\[ \text{But probably won’t take off points, still hard for us to read.} \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1 \);
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

Multiplicative inverse of \( x \) is \( y \) where \( xy = 1 \); 1 is multiplicative identity element.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

Multiplicative inverse of \( x \) is \( y \) where \( xy = 1 \); 1 is multiplicative identity element.

In modular arithmetic, 1 is the multiplicative identity element.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1; \)

1 is multiplicative identity element.

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \) **mod** \( m \) **is** \( y \) **with** \( xy = 1 \) (mod \( m \)).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \); 
**1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element. 
**Multiplicative inverse of** \( x \) **mod** \( m \) **is** \( y \) **with** \( xy = 1 \ (\text{mod} \ m) \). 
For 4 modulo 7 inverse is 2: 
\[ 2 \cdot 4 \equiv 8 \equiv 1 \ (\text{mod} \ 7). \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1 \);

\( 1 \) is multiplicative identity element.

In modular arithmetic, \( 1 \) is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) is \( y \) with \( xy = 1 \mod m \).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

Can solve \( 4x = 5 \mod 7 \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1; \)

1 **is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \mod m \).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

Can solve \( 4x = 5 \mod 7 \).

\[ 2 \cdot 4x = 2 \cdot 5 \mod 7 \]
Inverses and Factors.

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Can solve \( 4x = 5 \mod 7). \]
\[ 2 \cdot 4x = 2 \cdot 5 \mod 7 \]
\[ 8x = 10 \mod 7 \]
Inverses and Factors.

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\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

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**Multiplicative inverse of** \( x \) **mod** \( m \) **is** \( y \) **with** \( xy = 1 \) **(mod** \( m \)).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \) **(mod** \( 7 \)).

Can solve \( 4x = 5 \) **(mod** \( 7 )**.
\[ 2 \cdot 4x = 2 \cdot 5 \text{ (mod } 7) \]
\[ 8x = 10 \text{ (mod } 7) \]
\[ x = 3 \text{ (mod } 7) \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

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For 4 modulo 7 inverse is 2: \(2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}\).

Can solve \(4x = 5 \pmod{7}\).
\[
\begin{align*}
2 \cdot 4x &= 2 \cdot 5 \pmod{7} \\
8x &= 10 \pmod{7} \\
x &= 3 \pmod{7}
\end{align*}
\]

Check!
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \iff \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \iff x = \frac{3}{2}. \]

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For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \ (\text{mod } 7). \]

Can solve \( 4x = 5 \) (mod 7).
\[ 2 \cdot 4x = 2 \cdot 5 \ (\text{mod } 7) \]
\[ 8x = 10 \ (\text{mod } 7) \]
\[ x = 3 \ (\text{mod } 7) \]

Check! \( 4(3) = 12 = 5 \) (mod 7).
Inverses and Factors.

Division: multiply by multiplicative inverse.

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

Can solve \( 4x = 5 \mod 7 \).

\[ x = 3 \mod 7 \implies \text{Check!} \ 4(3) = 12 = 5 \mod 7. \]
Inverses and Factors.

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\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \) (**mod** 7).

Can solve \( 4x = 5 \) (**mod** 7).
\( x = 3 \) (**mod** 7) \( \implies \) Check! \( 4(3) = 12 = 5 \) (**mod** 7).

For 8 modulo 12: no multiplicative inverse!
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

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\( x = 3 \mod 7 \) \( \implies \) **Check!** \( 4(3) = 12 = 5 \mod 7 \).

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4”
Inverses and Factors.

Division: multiply by multiplicative inverse.

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For 4 modulo 7 inverse is 2: \(2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}\).

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For 8 modulo 12: no multiplicative inverse!

“Common factor of 4” \(\implies\)
\(8k - 12\ell\) is a multiple of four for any \(\ell\) and \(k\) \(\implies\)
Inverses and Factors.

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \) (mod 7).

Can solve \( 4x = 5 \) (mod 7).
\( x = 3 \) (mod 7) \( \iff \) Check! \( 4(3) = 12 = 5 \) (mod 7).

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4” \( \implies \)
\( 8k - 12\ell \) is a multiple of four for any \( \ell \) and \( k \) \( \implies \)
\( 8k \not\equiv 1 \) (mod 12) for any \( k \).
Mark true statements.
(A) Multiplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of \((n - 1) \mod n\) = \((n - 1) \mod n\).
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of 4 = \(-1 \mod 5\).
(E) \((-1) \times (-1) = 1\). Woohoo.
(F) Multiplicative inverse of 4 mod 5 is 4 mod 5.
Mark true statements.
(A) Multiplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of \((n - 1) \mod n = ((n - 1) \mod n)\).
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of 4 = \(-1 mod5\).
(E) \((-1) \times (-1) = 1\). Woohoo.
(F) Multiplicative inverse of 4 mod 5 is 4 mod 5.

(C) is false. 0.5 has no meaning in arithmetic modulo 5.
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, $\text{gcd}(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$. 
Greatest Common Divisor and Inverses.

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If greatest common divisor of \( x \) and \( m \), \( \gcd(x, m) \), is 1, then \( x \) has a multiplicative inverse modulo \( m \).

**Proof \( \iff \):**

**Claim:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).
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If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

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**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to one of $m$ equivalence classes modulo $m$. 

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\( \Longrightarrow \) One must correspond to 1 modulo \( m \). **Inverse Exists!**

Proof of Claim: If not distinct, then \( \exists a, b \in \{0, \ldots, m - 1\}, a \neq b \),
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, \( \gcd(x, m) \), is 1, then $x$ has a multiplicative inverse modulo $m$.

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\[ \implies \text{One must correspond to } 1 \mod m. \text{ Inverse Exists!} \]

Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m - 1\}, a \neq b$, where

\[ (ax \equiv bx \mod m) \implies (a - b)x \equiv 0 \mod m \]
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Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}$, $a \neq b$, where

$$(ax \equiv bx \pmod{m}) \implies (a - b)x \equiv 0 \pmod{m}$$

Or $(a - b)x = km$ for some integer $k$. 
Greatest Common Divisor and Inverses.

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If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

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$\gcd(x, m) = 1$
Greatest Common Divisor and Inverses.

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If greatest common divisor of $x$ and $m$, gcd$(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

**Proof $\iff$:**
**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to one of $m$ equivalence classes modulo $m$.
⇒ One must correspond to 1 modulo $m$. **Inverse Exists!**

Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}$, $a \neq b$, where

$$(ax \equiv bx \mod m) \iff (a - b)x \equiv 0 \mod m$$

Or $(a - b)x = km$ for some integer $k$.

$gcd(x, m) = 1$
⇒ Prime factorization of $m$ and $x$ do not contain common primes.
Greatest Common Divisor and Inverses.

Thm:
If greatest common divisor of \(x\) and \(m\), gcd\((x,m)\), is 1, then \(x\) has a multiplicative inverse modulo \(m\).

Proof \(\Rightarrow\):
Claim: The set \(S = \{0x, 1x, \ldots, (m-1)x\}\) contains \(y \equiv 1 \mod m\) if all distinct modulo \(m\).

Each of \(m\) numbers in \(S\) correspond to one of \(m\) equivalence classes modulo \(m\).
\(\Rightarrow\) One must correspond to 1 modulo \(m\). \textbf{Inverse Exists!}

Proof of Claim: If not distinct, then \(\exists a, b \in \{0, \ldots, m-1\}, a \neq b, \) where \(ax \equiv bx \mod m\) \(\Rightarrow (a - b)x \equiv 0 \mod m\)
Or \((a - b)x = km\) for some integer \(k\).

\(gcd(x, m) = 1\)
\(\Rightarrow\) Prime factorization of \(m\) and \(x\) do not contain common primes.
\(\Rightarrow\) \((a - b)\) factorization contains all primes in \(m\)'s factorization.
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

**Proof $\implies$ :**
**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to one of $m$ equivalence classes modulo $m$.

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Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}$, $a \neq b$, where

$(ax \equiv bx \mod m) \implies (a - b)x \equiv 0 \mod m$

Or $(a - b)x = km$ for some integer $k$.

$\gcd(x, m) = 1$

$\implies$ Prime factorization of $m$ and $x$ do not contain common primes.  
$\implies (a - b)$ factorization contains all primes in $m$'s factorization.  
So $(a - b)$ has to be multiple of $m$.  

Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of \( x \) and \( m \), \( \gcd(x, m) \), is 1, then \( x \) has a multiplicative inverse modulo \( m \).

**Proof \( \implies \):**

**Claim:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

Each of \( m \) numbers in \( S \) correspond to one of \( m \) equivalence classes modulo \( m \).

\( \implies \) One must correspond to 1 modulo \( m \). **Inverse Exists!**

Proof of Claim: If not distinct, then \( \exists a, b \in \{0, \ldots, m - 1\} \), \( a \neq b \), where

\[
(ax \equiv bx \mod m) \implies (a - b)x \equiv 0 \mod m
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Or \( (a - b)x = km \) for some integer \( k \).

\( \gcd(x, m) = 1 \)

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\( \implies (a - b) \) factorization contains all primes in \( m \)'s factorization.

So \( (a - b) \) has to be multiple of \( m \).

\( \implies (a - b) \geq m. \)
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of \( x \) and \( m \), \( \gcd(x, m) \), is 1, then \( x \) has a multiplicative inverse modulo \( m \).

**Proof \( \implies \):**

**Claim:** The set \( S = \{ 0x, 1x, \ldots, (m-1)x \} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

Each of \( m \) numbers in \( S \) correspond to one of \( m \) equivalence classes modulo \( m \).

\( \implies \) One must correspond to 1 modulo \( m \). **Inverse Exists!**

Proof of Claim: If not distinct, then \( \exists a, b \in \{0, \ldots, m-1\}, \ a \neq b \), where

\[
(ax \equiv bx \mod m) \implies (a - b)x \equiv 0 \mod m
\]

Or \((a - b)x = km\) for some integer \( k \).

\[\gcd(x, m) = 1\]

\( \implies \) Prime factorization of \( m \) and \( x \) do not contain common primes.

\( \implies \) \((a - b)\) factorization contains all primes in \( m \)'s factorization.

So \((a - b)\) has to be multiple of \( m \).

\( \implies \) \((a - b) \geq m\). But \( a, b \in \{0, \ldots m-1\} \).
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of \( x \) and \( m \), \( \gcd(x, m) \), is 1, then \( x \) has a multiplicative inverse modulo \( m \).

**Proof \( \Rightarrow \):**

**Claim:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

Each of \( m \) numbers in \( S \) correspond to one of \( m \) equivalence classes modulo \( m \).

\( \Rightarrow \) One must correspond to 1 modulo \( m \). **Inverse Exists!**

Proof of Claim: If not distinct, then \( \exists a, b \in \{0, \ldots, m - 1\}, a \neq b \), where

\[ (ax \equiv bx \pmod{m}) \Rightarrow (a - b)x \equiv 0 \pmod{m} \]

Or \( (a - b)x = km \) for some integer \( k \).

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So \( (a - b) \) has to be multiple of \( m \).

\( \Rightarrow (a - b) \geq m \). But \( a, b \in \{0, \ldots, m - 1\} \). Contradiction.
Greatest Common Divisor and Inverses.

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If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

**Proof $\implies$ :**

**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to one of $m$ equivalence classes modulo $m$.

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Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}, a \neq b$, where $(ax \equiv bx \mod m)) \implies (a - b)x \equiv 0 \mod m$

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$\gcd(x, m) = 1$

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So $(a - b)$ has to be multiple of $m$.

$\implies (a - b) \geq m$. But $a, b \in \{0, \ldots m-1\}$. Contradiction. \qed
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).
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... 

For $x = 4$ and $m = 6$. All products of 4...
Proof review. Consequence.

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\( S = \)
**Proof review. Consequence.**

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... For $x = 4$ and $m = 6$. All products of 4...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\}$
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... 

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} 
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing \( \mod 6 \)
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]

reducing \((\mod 6)\)

\[ S = \{0, 4, 2, 0, 4, 2\} \]
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Proof review. Consequence.

Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

\( S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \)

reducing \( \mod 6 \)

\( S = \{0, 4, 2, 0, 4, 2\} \)

Not distinct.
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]

reducing \( \pmod{6} \)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2.
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

... For $x = 4$ and $m = 6$. All products of 4...

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reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2. Can’t be 1.
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

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For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
\begin{align*}
S &= \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \\
&\text{reducing (mod 6)} \\
S &= \{0, 4, 2, 0, 4, 2\}
\end{align*}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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... 

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing \( \mod 6 \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\( S = \)
Thm: If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

Proof Sketch: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

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Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing \( \pmod{6} \)

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S = \{0, 4, 2, 0, 4, 2\}
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Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

... 

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]
reducing (mod 6)  
\[ S = \{0, 4, 2, 0, 4, 2\} \]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]
All distinct,
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... 

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing (mod 6)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1!
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]
reducing \( \mod 6 \)

\[ S = \{0, 4, 2, 0, 4, 2\} \]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]
All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

... For $x = 4$ and $m = 6$. All products of 4...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$ reducing (mod 6)

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Not distinct. Common factor 2. Can’t be 1. No inverse.

For $x = 5$ and $m = 6$.

$S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}$

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?)
**Proof review. Consequence.**

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

**...**

For $x = 4$ and $m = 6$. All products of 4...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$

reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2. Can’t be 1. No inverse.

For $x = 5$ and $m = 6$.

$S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}$

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...
\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]
reducing \( \mod 6 \)
\[
S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...
\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]
reducing \((\text{mod } 6)\)
\[
S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct, contains \( 1 \)! 5 is multiplicative inverse of 5 \((\text{mod } 6)\).

(Hmm. What normal number is it own multiplicative inverse?) \( 1 \) \(-1\).

\[
5x = 3 \pmod{6}
\]
Thm: If gcd\((x, m) = 1\), then \(x\) has a multiplicative inverse modulo \(m\).

Proof Sketch: The set \(S = \{0x, 1x, \ldots, (m - 1)x\}\) contains \(y \equiv 1 \mod m\) if all distinct modulo \(m\).

... For \(x = 4\) and \(m = 6\). All products of 4...
\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]
reducing \(\mod 6\)
\[
S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \(x = 5\) and \(m = 6\).
\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct, contains 1! 5 is multiplicative inverse of 5 \((\mod 6)\).
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \text{ What is } x?
\]
**Proof review. Consequence.**

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing \( \mod 6 \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \text{ What is } x? \text{ Multiply both sides by 5.}
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]

Reducing \( \mod 6 \)
\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \]

What is \( x \)? Multiply both sides by 5.
\[ x = 15 \]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

\[
\text{...}
\]

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing \((\mod 6)\)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

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S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1! 5 is multiplicative inverse of 5 \((\mod 6)\). (Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \text{ What is } x? \text{ Multiply both sides by 5.}
\]

\[
x = 15 = 3 \mod 6
\]
**Proof review. Consequence.**

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing (mod 6)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \quad \text{What is } x? \quad \text{Multiply both sides by 5.}
\]

\[
x = 15 = 3 \mod 6
\]

\[
4x = 3 \mod 6
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...

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reducing \( \pmod{6} \)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

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For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \pmod{6} \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \pmod{6} \] What is \( x \)? Multiply both sides by 5.

\[ x = 15 = 3 \pmod{6} \]

\[ 4x = 3 \pmod{6} \] No solutions.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \quad \text{What is } x? \quad \text{Multiply both sides by 5.}
\]

\[
x = 15 = 3 \mod 6
\]

4\(x = 3 \mod 6 \) No solutions. Can’t get an odd.
**Proof** review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]
reducing \((\bmod 6)\)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]
All distinct, contains 1! 5 is multiplicative inverse of 5 \((\bmod 6)\).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \pmod{6} \text{ What is } x? \text{ Multiply both sides by } 5.
\]

\[
x = 15 = 3 \pmod{6}
\]

\[
4x = 3 \pmod{6} \text{ No solutions. Can’t get an odd.}
\]

\[
4x = 2 \pmod{6}
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing \( \mod 6 \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \) What is \( x \)? Multiply both sides by 5.
\]

\[
x = 15 = 3 \mod 6
\]

\[
4x = 3 \mod 6 \) No solutions. Can’t get an odd.
\]

\[
4x = 2 \mod 6 \) Two solutions!
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**Proof review. Consequence.**

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\[ 5x = 3 \pmod{6} \] What is \( x \)? Multiply both sides by 5.

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Very different for elements with inverses.
Proof Review 2: Bijections.

If $\gcd(x,m) = 1$. 
Proof Review 2: Bijectons.

If \( \gcd(x,m) = 1 \).
Then the function \( f(a) = xa \mod m \) is a bijection.
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One to one: there is a unique pre-image.
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  Onto: the sizes of the domain and co-domain are the same.
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\( x = 3, \ m = 4. \)

\( f(1) = 3(1) = 3 \ (\mod 4), \)
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\( x = 3, m = 4 \).

\[ f(1) = 3(1) = 3 \mod 4, f(2) = 6 = 2 \mod 4, \]

\( f(0) = 0 \).

Not a bijection.
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Bijection $\equiv$ unique pre-image and same size.
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All the images are distinct. \( \implies \) unique pre-image for any image.
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$x = 2, m = 4$.

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Not a bijection.
Poll

Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
(B) $f(x) = ax \mod n$ for $x \in \{0, \ldots, n-1\}$ and $\gcd(a, n) = 2$
(C) $f(x) = ax \mod n$ for $x \in \{0, \ldots, n-1\}$ and $\gcd(a, n) = 1$
Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
(B) $f(x) = ax \mod n$ for $x \in \{0, \ldots, n-1\}$ and $\gcd(a, n) = 2$
(C) $f(x) = ax \mod n$ for $x \in \{0, \ldots, n-1\}$ and $\gcd(a, n) = 1$

(B) is not.
Only if

Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$. 
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Assume $a$ is $x^{-1}$, or $ax = 1 + km$. 
Only if

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$x = nd$ and $m = \ell d$ for $d > 1$. 
Only if

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Thus,

$a(nd) = 1 + k\ell d$ or $d(na - k\ell) = 1$. 
Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume $a$ is $x^{-1}$, or $ax = 1 + km$.

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Thus,

$$a(nd) = 1 + k \ell d$$
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$$d(na - k \ell) = 1.$$

But $d > 1$ and $n = (na - k \ell) \in \mathbb{Z}$. 
Only if

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$a(nd) = 1 + k\ell d$ or $d(na - k\ell) = 1$.

But $d > 1$ and $n = (na - k\ell) \in \mathbb{Z}$.

so $dn \neq 1$ and $dn = 1$. Contradiction.
Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).

Assume \( a \) is \( x^{-1} \), or \( ax = 1 + km \).

\( x = nd \) and \( m = \ell d \) for \( d > 1 \).

Thus,
\[
a(nd) = 1 + k\ell d \text{ or } d(na - k\ell) = 1.
\]

But \( d > 1 \) and \( n = (na - k\ell) \in \mathbb{Z} \).

so \( dn \neq 1 \) and \( dn = 1 \). Contradiction.
Finding inverses.

How to find the inverse?

Find $\gcd(x, m)$. Greater than 1? No multiplicative inverse. Equal to 1? Multiplicative inverse.

Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$. Very slow.
Finding inverses.

How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Find gcd $(x, m)$. 
Finding inverses.

How to find the inverse?
How to find if \( x \) has an inverse modulo \( m \)?
Find \( \text{gcd} \ (x, m) \).
  Greater than 1?
Finding inverses.

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- Greater than 1? No multiplicative inverse.
- Equal to 1?
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How to find \textbf{if} \(x\) has an inverse modulo \(m\)?

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Very slow.
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How to find if $x$ has an inverse modulo $m$?

Find $\text{gcd} \ (x, m)$.  
   Greater than 1? No multiplicative inverse.  
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Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$.  
Very slow.
Inverses

Next up.
Inverses

Next up.
Inverses

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Euclid’s Algorithm.
Inverses

Next up.

Euclid’s Algorithm.
Runtime.
Inverses

Next up.

Euclid’s Algorithm.
Runtime.
Euclid’s Extended Algorithm.
Does 2 have an inverse mod 8?

Any multiple of 2 is 2 away from 0 + 8k for any k ∈ N.

Does 2 have an inverse mod 9?

Yes. 5 * 2 = 10 = 1 mod 9.

Does 6 have an inverse mod 9?

No. Any multiple of 6 is 3 away from 0 + 9k for any k ∈ N.

3 = gcd(6, 9)!

x has an inverse modulo m if and only if gcd(x, m) = 1?

Yes.

Now what?: Compute gcd! Compute Inverse modulo m.
Does 2 have an inverse mod 8? No.

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Does 2 have an inverse mod 9? Yes.

5 \cdot 2 \equiv 1 \pmod{9}.

Does 6 have an inverse mod 9? No.

Any multiple of 6 is 3 away from 0 + 9k for any k ∈ N.

3 = \gcd(6, 9)!

x has an inverse modulo m if and only if \gcd(x, m) > 1?

No.

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Yes.

Now what?:

Compute gcd!

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$2(5) = 10 = 1 \mod 9$. 
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   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
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Does 2 have an inverse mod 8? No.  
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Does 6 have an inverse mod 9? No.  
Any multiple of 6 is 3 away from \(0 + 9k\) for any \(k \in \mathbb{N}\).  
\[3 = \gcd(6, 9)\]

\(x\) has an inverse modulo \(m\) if and only if  
\[\gcd(x, m) > 1?\]
Does 2 have an inverse mod 8? No.
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Now what?:
   Compute \( \gcd \)!
Does 2 have an inverse mod 8? No. Any multiple of 2 is 2 away from \( 0 + 8k \) for any \( k \in \mathbb{N} \).

Does 2 have an inverse mod 9? Yes. 5

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3 = \gcd(6, 9)!
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\( x \) has an inverse modulo \( m \) if and only if

\[
\gcd(x, m) > 1? \text{ No.}
\]

\[
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Now what?:

Compute gcd!

Compute Inverse modulo \( m \).
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Divisibility...

Notation: $d | x$ means “$d$ divides $x$” or
**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 

**Fact:** If $d|x$ and $d|y$ then $d|(x+y)$ and $d|(x-y)$. 

**Is it a fact?** Yes?

**Proof:** 
$d|x$ and $d|y$ or $x = ℓd$ and $y = kd =⇒ x − y = kd − ℓd = (k − ℓ)d =⇒ d|(x−y)$. 

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Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$. 
**Divisibility...**

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Fact:** If \( d \mid x \) and \( d \mid y \) then \( d \mid (x + y) \) and \( d \mid (x - y) \).

Is it a fact?
Notation: $d|\ x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|\ x$ and $d|\ y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes?
Notation: \(d \mid x\) means “\(d\) divides \(x\)” or \(x = kd\) for some integer \(k\).

Fact: If \(d \mid x\) and \(d \mid y\) then \(d \mid (x + y)\) and \(d \mid (x - y)\).

Is it a fact? Yes? No?
Notation: $d|x$ means “$d$ divides $x$” or 
\[ x = kd \] for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes? No?

Proof: $d|x$ and $d|y$ or
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes? No?

Proof: $d|x$ and $d|y$ or $x = ℓd$ and $y = kd$
Divisibility...

**Notation:** $d| x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Fact:** If $d| x$ and $d| y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes? No?

**Proof:** $d| x$ and $d| y$ or $x = ℓd$ and $y = kd$

$\implies x - y = kd - ℓd$
**Notation:** $d|\ x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Fact:** If $d|\ x$ and $d|\ y$ then $d|\ (x + y)$ and $d|\ (x - y)$.

Is it a fact? Yes? No?

**Proof:** $d|\ x$ and $d|\ y$ or

$x = \ell d$ and $y = kd$

$$
\implies x - y = kd - \ell d = (k - \ell)d
$$
Notation: $d|\ x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

Is it a fact? Yes? No?

Proof: $d|x$ and $d|y$ or $x = \ell d$ and $y = kd$

$\Rightarrow \ x - y = kd - \ell d = (k - \ell)d \ \Rightarrow \ d|(x - y)$
**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Fact:** If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$.

Is it a fact? Yes? No?

**Proof:** $d | x$ and $d | y$ or

$x = \ell d$ and $y = kd$

$\implies x - y = kd - \ell d = (k - \ell)d \implies d | (x - y)$
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod}(x, y) \).

**Proof:**
\[
\text{mod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y = x - \left\lfloor \frac{s}{d} \right\rfloor \cdot y = kd - s \ell d \text{ for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d = (k - s \ell) d
\]
Therefore \( d \mid \text{mod}(x, y) \). And \( d \mid y \) since it is in condition.

**Lemma 2:** If \( d \mid y \) and \( d \mid \text{mod}(x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof:** Similar.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \text{mod}(x, y)) \).

**Proof:** \( x \) and \( y \) have the same set of common divisors as \( x \) and \( \text{mod}(x, y) \) by Lemma 1 and 2. Same common divisors \( \Rightarrow \) largest is the same.
More divisibility

**Notation:** $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d \mid x$ and $d \mid y$ then $d \mid y$ and $d \mid \text{mod} (x, y)$.

**Lemma 2:** If $d \mid y$ and $d \mid \text{mod} (x, y)$ then $d \mid y$ and $d \mid x$.

**GCD Mod Corollary:** $\gcd (x, y) = \gcd (y, \text{mod} (x, y))$.

**Proof:** $x$ and $y$ have the same set of common divisors as $x$ and $\text{mod} (x, y)$ by Lemma 1 and 2. The same common divisors implies the largest is the same.
More divisibility

**Notation:** $d \mid x$ means “$d$ divides $x$” or 
$x = kd$ for some integer $k$.

**Lemma 1:** If $d \mid x$ and $d \mid y$ then $d \mid y$ and $d \mid \text{mod}(x, y)$.

**Proof:**

$\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y$
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod}(x, y)$.

**Proof:**

$$\text{mod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y$$

$$= x - [s] \cdot y$$

for integer $s$. 

**Lemma 2:**

If $d | y$ and $d | \text{mod}(x, y)$ then $d | y$ and $d | x$.

**Proof...:** Similar.

**GCD Mod Corollary:**

$$\gcd(x, y) = \gcd(y, \text{mod}(x, y))$$

**Proof:** $x$ and $y$ have the same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2. The same common divisors $\Rightarrow$ the largest is the same.
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \).

**Proof:**
\[
\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y = x - \lfloor s \rfloor \cdot y \quad \text{for integer } s = kd - sld \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or\n\[ x = kd \text{ for some integer } k. \]

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod}(x, y) \).

**Proof:**
\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \text{ for integer } s \\
= kd - s\ell d \text{ for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod}(x, y)$.

**Proof:**
\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]
Therefore $d | \text{mod}(x, y)$. 
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} \ (x, y)$.

**Proof:**

\[
\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s \ell) d
\]

Therefore $d | \text{mod} \ (x, y)$. And $d | y$ since it is in condition.
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d| \text{mod } (x, y)$.

**Proof:**

\[
\text{mod } (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]

Therefore $d| \text{mod } (x, y)$. And $d|y$ since it is in condition. \qed
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**

\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
\]

\[= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s\]

\[= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d\]

\[= (k - s\ell )d\]

Therefore $d | \text{mod} (x, y)$. And $d | y$ since it is in condition. $\square$

**Lemma 2:** If $d | y$ and $d | \text{mod} (x, y)$ then $d | y$ and $d | x$.

**Proof...:** Similar.
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x,y)$.

**Proof:**

\[
\text{mod}(x,y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]

Therefore $d|\text{mod}(x,y)$. And $d|y$ since it is in condition.

**Lemma 2:** If $d|y$ and $d|\text{mod}(x,y)$ then $d|y$ and $d|x$.

**Proof...:** Similar. Try this at home.
More divisibility

**Notation:** $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d \mid x$ and $d \mid y$ then $d \mid y$ and $d \mid \text{mod} \ (x, y)$.

**Proof:**
\[
\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - [s] \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]

Therefore $d \mid \text{mod} \ (x, y)$. And $d \mid y$ since it is in condition.

**Lemma 2:** If $d \mid y$ and $d \mid \text{mod} \ (x, y)$ then $d \mid y$ and $d \mid x$.

**Proof...:** Similar. Try this at home.  

\[\square\text{ish.}\]
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \).

**Proof:**

\[
\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]

Therefore \( d \mid \text{mod} \ (x, y) \). And \( d \mid y \) since it is in condition. \( \square \)

**Lemma 2:** If \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof...:** Similar. Try this at home. \( \square \)ish.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \text{ mod} \ (x, y)) \).
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{ mod } (x, y)$.

**Proof:**

\[
\begin{align*}
\text{mod } (x, y) &= x - \lfloor x/y \rfloor \cdot y \\
&= x - \lfloor s \rfloor \cdot y \quad \text{ for integer } s \\
&= kd - s \ell d \quad \text{ for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
&= (k - s \ell)d
\end{align*}
\]

Therefore $d|\text{ mod } (x, y)$. And $d|y$ since it is in condition.

**Lemma 2:** If $d|y$ and $d|\text{ mod } (x, y)$ then $d|y$ and $d|x$.

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \text{ mod } (x, y))$.

**Proof:** $x$ and $y$ have **same** set of common divisors as $x$ and $\text{ mod } (x, y)$ by Lemma 1 and 2.
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x, y)$.

**Proof:**
$$\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y$$
$$= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s$$
$$= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d$$
$$= (k - s\ell)d$$
Therefore $d|\text{mod}(x, y)$. And $d|y$ since it is in condition.

**Lemma 2:** If $d|y$ and $d|\text{mod}(x, y)$ then $d|y$ and $d|x$.
**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \text{mod}(x, y))$.
**Proof:** $x$ and $y$ have **same** set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2.
Same common divisors $\implies$ largest is the same.
More divisibility

**Notation:** \(d|x\) means “\(d\) divides \(x\)” or \(x = kd\) for some integer \(k\).

**Lemma 1:** If \(d|x\) and \(d|y\) then \(d|y\) and \(d|\ \text{mod} \ (x, y)\).

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - [s] \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]
Therefore \(d|\ \text{mod} \ (x, y)\). And \(d|y\) since it is in condition.

**Lemma 2:** If \(d|y\) and \(d|\ \text{mod} \ (x, y)\) then \(d|y\) and \(d|x\).

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** \(\gcd(x, y) = \gcd(y, \ \text{mod} \ (x, y))\).

**Proof:** \(x\) and \(y\) have **same** set of common divisors as \(x\) and \(\text{mod} \ (x, y)\) by Lemma 1 and 2.

Same common divisors \(\implies\) largest is the same.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \text{mod}(x, y)) \).
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)?
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \) ?  
7
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?
Euclid’s algorithm.

**GCD Mod Corollary:** $\text{gcd}(x, y) = \text{gcd}(y, \mod(x, y))$.

Hey, what’s $\text{gcd}(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\text{gcd}(x, 0)$? $x$
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? \( 7 \) since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))   ***
```
Euclid’s algorithm.

**GCD Mod Corollary:** $\text{gcd}(x, y) = \text{gcd}(y, \text{mod}(x, y))$.

Hey, what’s $\text{gcd}(7, 0)$? $7$ since 7 divides 7 and 7 divides 0
What’s $\text{gcd}(x, 0)$? $x$

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** $(\text{euclid } x \ y) = \text{gcd}(x, y)$ if $x \geq y$. 
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))) ***

**Theorem:** \( (euclid x y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? x

\[
\text{(define (euclid x y)}
\begin{align*}
&\quad \text{(if (= y 0)} \\
&\quad \quad x \\
&\quad \quad (\text{euclid y (mod x y)}))) \quad ***
\end{align*}
\]

**Theorem:** \( (\text{euclid x y)} = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0, \ “x \text{ divides } y \text{ and } x” \)
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```
(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y))))
```  

***

**Theorem:** \( (\text{euclid } x \ y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies \text{“} x \text{ is common divisor and clearly largest.”} \]
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))) ***
```

**Theorem:** \( (\text{euclid } x y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies \text{“} x \text{ is common divisor and clearly largest.”} \]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y)))) ***

**Theorem:** $(\text{euclid } x \ y) = \gcd(x, y)$ if $x \geq y$.

**Proof:** Use Strong Induction.

**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”
  $\implies$ “$x$ is common divisor and clearly largest.”

**Induction Step:** $\mod(x, y) < y \leq x$ when $x \geq y$

call in line (***)) meets conditions plus arguments “smaller”
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))) ***

**Theorem:** $(\text{euclid } x \ y) = \gcd(x, y)$ if $x \geq y$.

**Proof:** Use Strong Induction.
**Base Case:** $y = 0$, "$x$ divides $y$ and $x"$
  $\implies$ "$x$ is common divisor and clearly largest."

**Induction Step:** $\mod(x, y) < y \leq x$ when $x \geq y$

call in line (***)) meets conditions plus arguments “smaller”
  and by strong induction hypothesis
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod(x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)? \( x \)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))
  ) ***

**Theorem:** \( (\text{euclid } x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\( \rightarrow \) “\( x \) is common divisor and clearly largest.”

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)

call in line (***), meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes \( \text{gcd}(y, \mod(x, y)) \)
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))
```

**Theorem:** \( (\text{euclid } x y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
\[ \implies \text{“} x \text{is common divisor and clearly largest.”} \]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)

call in line (***)) meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes \( \gcd(y, \mod(x, y)) \)
which is \( \gcd(x, y) \) by GCD Mod Corollary.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \ mod \ (x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))
```

**Theorem:** \((\text{euclid } x \ y) = \gcd(x, y)\) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), "\( x \) divides \( y \) and \( x \)"

\[ \implies \text{"} x \text{ is common divisor and clearly largest."} \]

**Induction Step:** \( \mod (x, y) < y \leq x \) when \( x \geq y \)

call in line (***) meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes \( \gcd(y, \ \mod (x, y)) \)
which is \( \gcd(x, y) \) by GCD Mod Corollary.
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: \( x \equiv y \pmod{N} \) if \( x = y + kN \) for some integer \( k \).
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For \(a \equiv b \pmod{N}\), and \(c \equiv d \pmod{N}\),
\[ac = bd \pmod{N}\] and \(a + b = c + d \pmod{N}\).
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Division?
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Fast cuz value drops by a factor of two every two recursive calls.
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Know if there is an inverse, but how do we find it?
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