Lecture 7. Outline.

1. Modular Arithmetic.

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1. Modular Arithmetic.
   Clock Math!!!

Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!
2. Inverses for Modular Arithmetic: Greatest Common Divisor.

Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!
2. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!

Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!
2. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!
3. Euclid's GCD Algorithm.

Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!
2. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!
3. Euclid's GCD Algorithm.
   A little tricky here!

Clock Math

If it is 1:00 now.

Clock Math

If it is 1:00 now.
What time is it in 2 hours?

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

Clock Math

If it is 1:00 now.
- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00!

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

5 is the same as 101 for a 12 hour clock system.

Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$
(Almost remainder, except for 12 and 0 are equivalent.)

Day of the week.

Today is Monday.

Day of the week.

Today is Monday.
What day is it a year from now?

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Day of the week.

Today is Monday.
What day is it a year from now? on February 9, 2016?
Number days.

Day of the week.

Today is Monday.
  What day is it a year from now? on February 9, 2016?
    Number days.
      0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

Day of the week.

Today is Monday.

- What day is it a year from now? on February 9, 2016?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

- 5 days from now.

Day of the week.

Today is Monday.
What day is it a year from now? on February 9, 2016?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7

Day of the week.

Today is Monday.

- What day is it a year from now? on February 9, 2016?
  - Number days.
    - 0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 2.

- 5 days from now. day 7 or day 0

Day of the week.

Today is Monday.

- What day is it a year from now? on February 9, 2016?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

- 5 days from now. day 7 or day 0 or Sunday.

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now.

Day of the week.

Today is Monday.
  What day is it a year from now? on February 9, 2016?
    Number days.
      0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
  5 days from now. day 7 or day 0 or Sunday.
  25 days from now. day 27

Day of the week.

Today is Monday.
What day is it a year from now? on February 9, 2016?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6.

Day of the week.

Today is Monday.
What day is it a year from now? on February 9, 2016?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6.
two days are equivalent up to addition/subtraction of multiple of 7.

Day of the week.

Today is Monday.
  What day is it a year from now? on February 9, 2016?
    Number days.
      0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
  5 days from now. day 7 or day 0 or Sunday.
  25 days from now. day 27 or day 6.
    two days are equivalent up to addition/subtraction of multiple of 7.
    11 days from now

Day of the week.

Today is Monday.
- What day is it a year from now? on February 9, 2016?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
- 5 days from now. day 7 or day 0 or Sunday.
- 25 days from now. day 27 or day 6.
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from now is day 6

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now. day 27 or day 6.

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from now is day 6 which is Saturday!

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now. day 27 or day 6.

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from now is day 6 which is Saturday!

What day is it a year from now?

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now. day 27 or day 6.

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from now is day 6 which is Saturday!

What day is it a year from now?

This year is not a leap year.

Day of the week.

Today is Monday.

  What day is it a year from now? on February 9, 2016?

    Number days.

      0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

  5 days from now. day 7 or day 0 or Sunday.

  25 days from now. day 27 or day 6.

    two days are equivalent up to addition/subtraction of multiple of 7.

    11 days from now is day 6 which is Saturday!

What day is it a year from now?

  This year is not a leap year. So 365 days from now.

Day of the week.

Today is Monday.
- What day is it a year from now? on February 9, 2016?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
- 5 days from now. day 7 or day 0 or Sunday.
- 25 days from now. day 27 or day 6.
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from now is day 6 which is Saturday!

What day is it a year from now?
- This year is not a leap year. So 365 days from now.
- Day 2+365 or day 367.

Day of the week.

Today is Monday.
What day is it a year from now? on February 9, 2016?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6.
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:

Day of the week.

Today is Monday.
- What day is it a year from now? on February 9, 2016?
  - Number days.
    - 0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 2.
- 5 days from now. day 7 or day 0 or Sunday.
- 25 days from now. day 27 or day 6.
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from now is day 6 which is Saturday!

What day is it a year from now?
- This year is not a leap year. So 365 days from now.
- Day 2+365 or day 367.

Smallest representation:
- subtract 7 until smaller than 7.

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now. day 27 or day 6.

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from now is day 6 which is Saturday!

What day is it a year from now?

This year is not a leap year. So 365 days from now.

Day 2+365 or day 367.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

Day of the week.

Today is Monday.
What day is it a year from now? on February 9, 2016?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6.
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
367/7

Day of the week.

Today is Monday.
What day is it a year from now? on February 9, 2016?
Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 2.
5 days from now. day 7 or day 0 or Sunday.
25 days from now. day 27 or day 6.
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from now is day 6 which is Saturday!

What day is it a year from now?
This year is not a leap year. So 365 days from now.
Day 2+365 or day 367.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
367/7 leaves quotient of 52 and remainder 3.

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now. day 27 or day 6.

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from now is day 6 which is Saturday!

What day is it a year from now?

This year is not a leap year. So 365 days from now.

Day 2+365 or day 367.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

367/7 leaves quotient of 52 and remainder 3.

or February 7, 2018 is a Wednesday.

Day of the week.

Today is Monday.

What day is it a year from now? on February 9, 2016?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 2.

5 days from now. day 7 or day 0 or Sunday.

25 days from now. day 27 or day 6.

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from now is day 6 which is Saturday!

What day is it a year from now?

This year is not a leap year. So 365 days from now.

Day 2+365 or day 367.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

367/7 leaves quotient of 52 and remainder 3.

or February 7, 2018 is a Wednesday.

Years and years...

80 years from now?

Years and years...

80 years from now? 20 leap years.

Years and years...

80 years from now? 20 leap years. 366×20 days

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7?

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7?

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60$

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$

Years and years...

80 years from now? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 2.

It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 2.

Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$

Remainder when dividing by 7?

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7$

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.
Or February 7, 2096 is Thursday!

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.
Or February 7, 2096 is Thursday!

Further Simplify Calculation:

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.
Or February 7, 2096 is Thursday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.
Or February 7, 2096 is Thursday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.
Or February 7, 2096 is Thursday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.
Or February 7, 2096 is Thursday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
Or Day 4.

Years and years...

80 years from now? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 2.
It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 2.
Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$
Remainder when dividing by 7? $102 = 14 \times 7 + 4$.
Or February 7, 2096 is Thursday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
Or Day 4. February 9, 2095 is Thursday.

Years and years...

80 years from now? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 2.

It is day $2 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 2.

Get Day: $2 + 2 \times 20 + 1 \times 60 = 102$

Remainder when dividing by 7? $102 = 14 \times 7 + 4$.

Or February 7, 2096 is Thursday!

Further Simplify Calculation:

20 has remainder 6 when divided by 7.

60 has remainder 4 when divided by 7.

Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.

Or Day 4. February 9, 2095 is Thursday.

“Reduce” at any time in calculation!

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.

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Mod 7 equivalence classes:

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$\{\ldots, -7, 0, 7, 14, \ldots\}$

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$\implies a + b \equiv c + d \pmod{m}$. □

Can calculate with representative in $\{0, \ldots, m - 1\}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

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$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12$

Notation

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$\mod(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12$

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$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = 4$

Notation

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \not{4} = 5$

Notation

$x \pmod{m}$ or $\bmod(x, m)$

- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\bmod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\bmod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 =$ ~~4~~ $= 5$

Work in this system.

Notation

$x \pmod{m}$ or $\mod(x, m)$
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Work in this system.
$a \equiv b \pmod{m}$.

Notation

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Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

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Modulus is m

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Modulus is m

$6 \equiv$

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Modulus is m

$6 \equiv 3 + 3$

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Work in this system.
$a \equiv b \pmod{m}$.
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Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10$

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Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

Notation

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$6 =$

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$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$.

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$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$.
But ok, if you really want.

Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies (\frac{1}{2}) \cdot 2x = (\frac{1}{2}) \cdot 3 \implies x = \frac{3}{2}.$$

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Can solve $4x = 5 \pmod{7}$.

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$S = \{0, 4, 2, 0, 4, 2\}$
Not distinct. Common factor 2.

For $x = 5$ and $m = 6$.
$S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}$
All distinct, contains 1!

Proof review. Consequence.

Thm: If $\gcd(x,m) = 1$, then x has a multiplicative inverse modulo m.

Proof Sketch: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.

... □

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$x = 15$

Proof review. Consequence.

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Very different for elements with inverses.

Proof Review 2: Bijections.

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Onto: the sizes of the domain and co-domain are the same.

$x = 3, m = 4.$

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$x = 2, m = 4.$
$f(1) = 2, f(2) = 0, f(3) = 2$

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Not a bijection.

Finding inverses.

How to find the inverse?

Finding inverses.

How to find the inverse?

How to find **if** x has an inverse modulo m?

Finding inverses.

How to find the inverse?

How to find **if** x has an inverse modulo m?

Find gcd (x, m).

Finding inverses.

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How to find **if** x has an inverse modulo m?

Find gcd (x, m).
Greater than 1?

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How to find **if** x has an inverse modulo m?

Find gcd (x, m).
Greater than 1? No multiplicative inverse.

Finding inverses.

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Find gcd (x, m).
  Greater than 1? No multiplicative inverse.
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Algorithm:

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Algorithm: Try all numbers up to x to see if it divides both x and m.

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Very slow.

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Inverses

Next up.

Inverses

Next up.

Inverses

Next up.

Euclid's Algorithm.

Inverses

Next up.

Euclid's Algorithm.
Runtime.

Inverses

Next up.

Euclid's Algorithm.
Runtime.
Euclid's Extended Algorithm.

Refresh

Does 2 have an inverse mod 8?

Refresh

Does 2 have an inverse mod 8? No.

Refresh

Does 2 have an inverse mod 8? No.

Any multiple of 2 is 2 away from $0+8k$ for any $k \in \mathbb{N}$.

Refresh

Does 2 have an inverse mod 8? No.

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Does 2 have an inverse mod 9?

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Does 2 have an inverse mod 8? No.

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Refresh

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Any multiple of 6 is 3 away from $0+9k$ for any $k \in \mathbb{N}$.

Refresh

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   $3 = gcd(6,9)$!

Refresh

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x has an inverse modulo m if and only if

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x has an inverse modulo m if and only if
$gcd(x,m) > 1$?

Refresh

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Excursion: Value and Size.

Before discussing running time of gcd procedure...

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What is the value of 1,000,000?

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one million or 1,000,000!

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Number of bits: 21.

Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?

one million or 1,000,000!

What is the "size" of 1,000,000?

Number of digits: 7.

Number of bits: 21.

For a number x, what is its size in bits?

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Euclid procedure is fast.

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2^{n-1} divisions! Exponential dependence on size!

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$2n$ is much faster! .. roughly 200 divisions.

Algorithms at work.

Trying everything

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

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euclid(700,568)
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  euclid(568, 132)
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Algorithms at work.

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Notice: The first argument decreases rapidly.

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Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

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```

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

(The second is less than the first.)

Proof.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Theorem: (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

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After $2 \log_2 x = O(n)$ recursive calls, argument x is 1 bit number.

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Proof of Fact: Recall that first argument decreases every call.

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$\implies$ true in one recursive call;

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Case 2: Will show “$y \geq x/2$” $\implies$ “$mod(x,y) \leq x/2$.”

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mod (x, y) is second argument in next recursive call,
and becomes the first argument in the next one.

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When $y \geq x/2$, then

$$\lfloor \frac{x}{y} \rfloor = 1,$$

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□

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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Computes the gcd(x, y) in $O(n)$ divisions.

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```

Computes the gcd(x, y) in $O(n)$ divisions.

For x and m, if gcd$(x, m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

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$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

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$$(3)12 + (-1)35 = 1.$$

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$(3)12 + (-1)35 = 1$.

$a = 3$ and $b = -1$.

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$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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        1
```

How did gcd get 11 from 35 and 12?

Make *d* out of *x* and *y*..?

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make *d* out of *x* and *y*..?

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make d out of x and y..?

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gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
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How did gcd get 11 from 35 and 12?
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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Make *d* out of *x* and *y*..?

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gcd(35,12)
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
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```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.

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Example:

```
ext-gcd(35,12)
```

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```
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  if y = 0 then return(x, 1, 0)
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Example:

```
ext-gcd(35,12)
  ext-gcd(12, 11)
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
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Example:

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ext-gcd(35,12)
  ext-gcd(12, 11)
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      ext-gcd(1,0)
```

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         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b1 = \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax+by$.
Example:

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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         (d, a, b) := ext-gcd(y, mod(x,y))
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Extended GCD Algorithm.

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ext-gcd(x,y)
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         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Theorem: Returns (d, a, b), where $d = gcd(a, b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x,0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x,0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**$(y, \mod (x,y))$ returns (d, a, b) with
$d = ay + b(\mod (x,y))$

[1] Assume d is $gcd(x,y)$ by previous proof.

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Proof: Strong Induction.[1]
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ext-gcd(x,y) calls **ext-gcd**($y, \mod(x,y)$) so

[1] Assume d is $gcd(x,y)$ by previous proof.

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Proof: Strong Induction.[1]
Base: ext-gcd($x, 0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

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Ind hyp: **ext-gcd**(y, $\mod(x, y)$) returns (d, a, b) with
$d = ay + b(\mod(x, y))$

ext-gcd(x, y) calls **ext-gcd**(y, $\mod(x, y)$) so

$$d = ay + b \cdot (\mod(x, y))$$

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x,0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**(y, $\bmod (x,y)$) returns (d,a,b) with
$d = ay + b(\bmod (x,y))$

ext-gcd(x,y) calls **ext-gcd**(y, $\bmod (x,y)$) so

$$\begin{aligned} d &= ay + b \cdot (\bmod (x,y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \end{aligned}$$

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Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x,0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**(y, mod (x,y)) returns (d,a,b) with
$d = ay + b(\bmod (x,y))$

ext-gcd(x,y) calls **ext-gcd**(y, mod (x,y)) so

$$\begin{aligned} d &= ay + b \cdot (\bmod (x,y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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ext-gcd(x,y) calls **ext-gcd**$(y, \mod(x,y))$ so

$$\begin{aligned} d &= ay + b\cdot(\mod(x,y)) \\ &= ay + b\cdot(x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!

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Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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Ind hyp: **ext-gcd**$(y, \mod(x,y))$ returns (d, a, b) with
$d = ay + b(\mod(x,y))$

ext-gcd(x,y) calls **ext-gcd**$(y, \mod(x,y))$ so

$$
\begin{aligned}
d &= ay + b\cdot(\mod(x,y)) \\
&= ay + b\cdot(x - \lfloor \frac{x}{y} \rfloor y) \\
&= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y
\end{aligned}
$$

And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds! □

[1] Assume d is $gcd(x,y)$ by previous proof.

Review Proof: step.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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```

Review Proof: step.

```
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```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)$

Review Proof: step.

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ext-gcd(x,y)
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         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

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```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

Returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$.

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Very different from elementary school: try 1, try 2, try 3...

Wrap-up

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$2^{n/2}$

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?

Wrap-up

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?
≤ 80 divisions.

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≤ 80 divisions.
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Internet Security.

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Internet Security.
Public Key Cryptography: 512 digits.

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512 divisions vs.

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Internet Security: Next Week!