Administrivia

Course evaluations at 26.05%. If they hit 30%, everyone gets an additional homework drop.
Recap

**Theorem (Markov's Inequality)** For any nonnegative random variable \( X \) and \( a > 0 \),
\[
P(X > a) \leq \frac{E[X]}{a}
\]

**Theorem (Chebyshev's Inequality)** For any random variable \( X \) with expectation \( \mu \) and variance \( \sigma^2 < \infty \),
\[
P(|X - \mu| \geq c) \leq \frac{\text{Var}[X]}{c^2}
\]
[\( X \) is more than a distance of \( c \) from its mean]

Note: Normal Random Variables
For \( X \sim \text{Normal}(\mu, \sigma^2) \) and \( Z \sim \text{Normal}(0, 1) \),
\[
\frac{X - \mu}{\sigma}
\]
The density of \( Z \) is symmetric \((\phi(z) = \phi(-z))\), so

For \( x_1, \ldots, x_n \) iid with mean \( \mu \) and variance \( \sigma^2 \),
Estimation

Def: An estimator is a
The bias of an estimator is
\[ \text{Bias}[\hat{X}] = \]
We say an estimator is unbiased if

Ex: Suppose \( X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p) \) for unknown parameter \( p \). Construct an unbiased estimator for \( p \).

Ex: Suppose \( X_1, \ldots, X_n \) are iid with unknown expectation \( \mu \) and variance \( \sigma^2 \). Construct an unbiased estimator for \( \mu \).

Note: Generally, for an estimator \( \hat{X} \) of \( \theta \), we want that
Chebyshev Confidence Intervals

Def: For $0 < \delta < 1$, a $(1-\delta)$ confidence interval for a fixed parameter $\theta$ is

Q. We flip a biased coin that flips heads with probability $p$ $n$ times. Let $X_1, \ldots, X_n$ be the results of the flips. Construct an unbiased estimator for $p$.

Construct a $(1-\delta)$ confidence interval for $p$. 
Chebyshev Confidence Intervals II

Q We flip a biased coin that flips heads with probability $p$ $n$ times. Let $X_1, \ldots, X_n$ be the results of the flips.
Suppose $n=1000$ and 120 of the flips are heads. Construct the 95% confidence interval.

Q Suppose $X_1, \ldots, X_n$ are iid with unknown expectation $\mu$ and known variance $\sigma^2 = 3$.
Find $n$ such that a 98% confidence interval has error at most 0.01.
Normal Confidence Intervals 1

Let \( X_1, \ldots, X_n \) be iid with expectation \( \mu \) and variance \( \sigma^2 \in (0, \infty) \). What is the approximate distribution of \( \bar{X} = \frac{1}{n} \sum X_i \) for large \( n^2 \)?

Suppose \( n \) is large. Construct a \((1-\delta)\) confidence interval for \( \mu \), the population mean.

Note: When the sample size is large, the sample standard deviation is a good approximation for \( \sigma \).
Normal Confidence Intervals 1

Q: We flip a biased coin that flips heads with probability $p$ $n$ times. Let $X_1, ..., X_n$ be the results of the flips. Suppose $n=1000$ and 120 of the flips are heads. Construct the 95% confidence interval.
Law of Large Numbers

Note: We have seen that the variance in $\bar{X}$ decreases as the sample size increases.

The (Law of Large Numbers) Let $X_1, \ldots, X_n$ be iid with expectation $\mu < \infty$. Let the sample mean

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

For any $\epsilon > 0$,

$$P(|\bar{X} - \mu| \leq \epsilon) \to 1 \text{ as } n \to \infty$$

Ex: I flip a biased coin with unknown probability $p$ of heads.
Consider the distribution of $\bar{X}$ for various values of $n$.

[Demo]