Communicating with Errors

We learned how to *encrypt* communication so that an eavesdropper cannot find out your personal information.

What if your enemy is not an eavesdropper, but *nature*?

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Today: We finish modular arithmetic and learn about polynomials.

Composite Moduli

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So, we have $24 = (4 \text{ in } \mathbb{Z}/5\mathbb{Z}, 3 \text{ in } \mathbb{Z}/7\mathbb{Z})$.

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- From $(4,3)$, can we go back to 24?

Solving Modular Congruences

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Does a solution always exist?

Chinese Remainder Theorem

Idea: Construct numbers Δ_1 and Δ_2 so that:

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- Similarly, $\Delta_2 = 5 \cdot (5^{-1} \bmod 7) = 15$.

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- So, $x = 4 \cdot 21 + 3 \cdot 15 = 129 \ldots$

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- So, $x = 4 \cdot 21 + 3 \cdot 15 = 129 \ldots$ which equals 24, modulo 35.

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- Similarly, $\Delta_2 = 5 \cdot (5^{-1} \bmod 7) = 15$.
- So, $x = 4 \cdot 21 + 3 \cdot 15 = 129 \ldots$ which equals 24, modulo 35.

This requires $\gcd(5, 7) = 1$.

Chinese Remainder Theorem

Chinese Remainder Theorem (CRT): If $y_1, \ldots, y_n$ are fixed numbers and the moduli $m_1, \ldots, m_n$ are pairwise coprime (i.e., $\gcd(m_i, m_j) = 1$ for all $i \neq j$), then the system

$$\begin{aligned} x &\equiv y_1 \pmod{m_1} \\ &\vdots \\ x &\equiv y_n \pmod{m_n} \end{aligned}$$

has a unique solution in $\mathbb{Z}/m_1 \cdots m_n\mathbb{Z}$.[1]

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- Why is the solution unique? Consider the map

  $$f : \mathbb{Z}/m_1 \cdots m_n\mathbb{Z} \to (\mathbb{Z}/m_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_n\mathbb{Z})$$

  given by $f(x) = (x \bmod m_1, \ldots, x \bmod m_n)$.

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  given by $f(x) = (x \bmod m_1, \ldots, x \bmod m_n)$.
- The CRT says that the map is surjective. But the domain and range are the same size—f is a bijection.

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Isomorphism

For pairs $(a_1, b_1), (a_2, b_2) \in (\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z})$, where $\gcd(m_1, m_2) = 1$, define addition and multiplication:

$$\begin{aligned}(a_1, b_1) + (a_2, b_2) &:= (a_1 + a_2 \bmod m_1,\ b_1 + b_2 \bmod m_2),\\ (a_1, b_1)(a_2, b_2) &:= (a_1 a_2 \bmod m_1,\ b_1 b_2 \bmod m_2).\end{aligned}$$

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For pairs $(a_1,b_1),(a_2,b_2) \in (\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z})$, where $\gcd(m_1,m_2) = 1$, define addition and multiplication:

$$(a_1,b_1)+(a_2,b_2) := (a_1+a_2 \bmod m_1,\ b_1+b_2 \bmod m_2),$$
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Consider the map f (the CRT map).

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Consider the map f (the CRT map). Then, for $x, y \in \mathbb{Z}/m_1 m_2\mathbb{Z}$,

$$\begin{aligned} f(x+y) &= (x + y \bmod m_1,\ x + y \bmod m_2) \\ &= (x \bmod m_1,\ x \bmod m_2) + (y \bmod m_1,\ y \bmod m_2) \\ &= f(x) + f(y). \end{aligned}$$

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What does this say?

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What does this say?

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- Convert x and y to $(\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z})$, then add them as pairs.

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- Convert x and y to $(\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z})$, then add them as pairs. We get $f(x) + f(y)$.

Isomorphism

We showed: $f(x+y) = f(x) + f(y)$.

[2]To learn more about this, take Math 113.

Isomorphism

We showed: $f(x+y) = f(x) + f(y)$. Similarly, it holds that $f(xy) = f(x)f(y)$.

$$\begin{aligned} f(xy) &= (xy \bmod m_1,\ xy \bmod m_2) \\ &= (x \bmod m_1,\ x \bmod m_2)(y \bmod m_1,\ y \bmod m_2) = f(x)f(y). \end{aligned}$$

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It does not really matter whether you do addition/multiplication in $\mathbb{Z}/m_1 m_2\mathbb{Z}$, or $(\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z})$.

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In general, $a^k \equiv a^{k \bmod \varphi(m)} \pmod{m}$, if $\gcd(a, m) = 1$.

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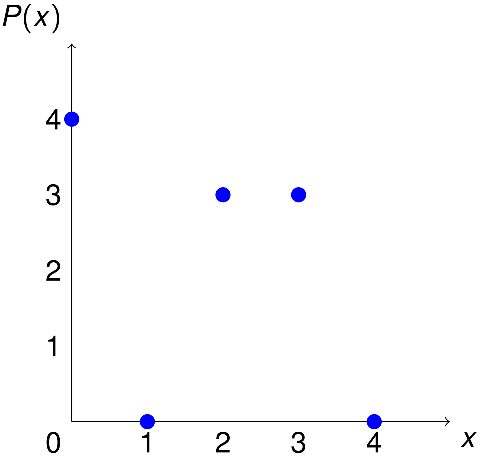

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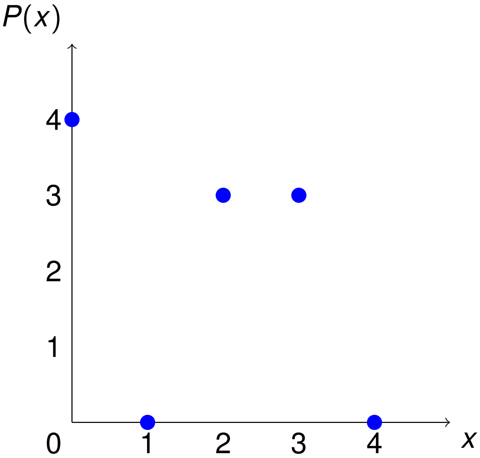

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- a set with two operations, $+$ (addition) and $\cdot$ (multiplication)
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- multiplication distributes over addition;
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What is not a field?

- $\mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z}$ for m composite: missing multiplicative inverses.

Polynomial Long Division

Recall the Division Algorithm: Given $a, b \in \mathbb{Z}$ with $b > 0$, then there exist unique $q \in \mathbb{Z}$ and $r \in \{0, 1, \ldots, b-1\}$ with $a = qb + r$.

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The algorithm needs multiplicative inverses—work in a field.

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- Try solving this system with linear algebra.

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$$x \equiv y_1 \pmod{m_1}$$
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find Δ_1 and Δ_2 so that

$$\Delta_1 \equiv 1 \pmod{m_1} \qquad \Delta_2 \equiv 0 \pmod{m_1}$$
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and then take $x = y_1\Delta_1 + y_2\Delta_2$.

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$$\Delta_i(x_j) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$$

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and then $P(x) = \sum_{i=1}^{d+1} y_i \Delta_i(x)$.

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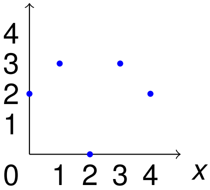

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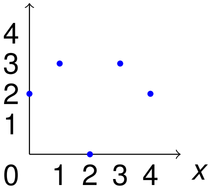
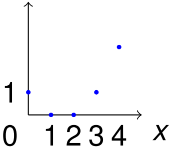
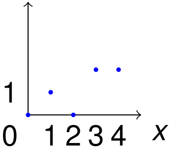


Here, $P(x) = 2 \cdot \Delta_1(x) + 3 \cdot \Delta_2(x)$.

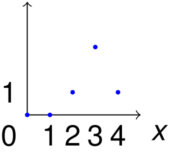
$\Delta_1(x) = 3x^2 + x + 1$



$\Delta_2(x) = 4x^2 + 2x$



$\Delta_3(x) = 3x^2 + 2x$



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For distinct points $x_1, \ldots, x_{d+1}$, construct:

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Slogan: $d + 1$ points uniquely determine a degree $\leq d$ polynomial.

Summary

- CRT: $\mathbb{Z}/m_1 \cdots m_n\mathbb{Z}$ and $(\mathbb{Z}/m_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_n\mathbb{Z})$ are isomorphic if $m_1, \ldots, m_n$ are pairwise coprime.
- If $\gcd(m_1, m_2) = 1$, then $\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2)$ (φ is multiplicative).
- Thus, $\varphi(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$ for a prime factorization.
- We work over fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/p\mathbb{Z}$ (AKA GF(p)) for p prime.
- A polynomial has a root a if and only if $P(x) = (x - a)Q(x)$ for some polynomial Q.
- A polynomial of degree d has at most d roots.
- Lagrange Interpolation: $d + 1$ distinct points uniquely determine a degree $\leq d$ polynomial.