Modular Arithmetic

Inverses.

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Euclid's Algorithm

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.

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Example: $365 \equiv 1 \pmod{7}$.

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Next year its 1 day later!

Notation

$x \pmod{m}$ or $\mod(x, m)$

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Work in this system.

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Work in this system.
$a \equiv b \pmod{m}$.

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$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Notation

$x \pmod{m}$ or $\bmod(x, m)$

- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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Modulus is m

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$6 \equiv$

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$a \equiv b \pmod{m}$.
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Modulus is m

$6 \equiv 3 + 3$

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Modulus is m

$6 \equiv 3+3 \equiv 3+10 \pmod{7}$.

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Generally, not $6 \pmod{7} = 13 \pmod{7}$.

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But ok, if you really want.

Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies (\frac{1}{2}) \cdot 2x = (\frac{1}{2}) \cdot 3 \implies x = \frac{3}{2}.$$

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$x = 3 \pmod{7}$
Check!

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Check! $4(3) = 12 = 5 \pmod{7}$.

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For 8 modulo 12: no multiplicative inverse!

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"Common factor of 4"

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"Common factor of 4" $\implies$
$8k - 12\ell$ is a multiple of four for any ℓ and k $\implies$

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"Common factor of 4" $\implies$
$8k - 12\ell$ is a multiple of four for any ℓ and k $\implies$
$8k \not\equiv 1 \pmod{12}$ for any k.

Greatest Common Divisor and Inverses.

Thm:
If **greatest common divisor** of x and m, $\gcd(x, m)$, is 1, then x has a multiplicative inverse modulo m.

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Pigenhole principle: Each of m numbers in S correspond to different one of m equivalence classes modulo m.
$\implies$ One must correspond to 1 modulo m.

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Thm:
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$5x = 3 \pmod 6$ What is x? Multiply both sides by 5.

$x = 15 = 3 \pmod 6$

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Proof review. Consequence.

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Proof Sketch: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.

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For $x = 4$ and $m = 6$. All products of 4...
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Very different for elements with inverses.

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Refresh

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$$\mod(x,y) = x - \lfloor x/y \rfloor \cdot y$$

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Excursion: Value and Size.

Before discussing running time of gcd procedure...

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one million or 1,000,000!

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Algorithms at work.

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euclid(700,568)
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"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
```

Algorithms at work.

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          euclid(4, 0)
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          euclid(4, 0)
            4
```

Algorithms at work.

Trying everything

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          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.

Algorithms at work.

Trying everything

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          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.
 At least a factor of 2 in two recursive calls.

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 ..., check $y/2$.

"(gcd x y)" at work.

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euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

(The second is less than the first.)

Proof.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Theorem: (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

Proof.

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Fact:
First arg decreases by at least factor of two in two recursive calls.

After $2\log_2 x = O(n)$ recursive calls, argument x is 1 bit number.

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1 division per recursive call.

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One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions. $\square$

Proof.

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Fact:
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Proof of Fact: Recall that first argument decreases every call.

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Case 1: $y < x/2$, first argument is y
$\implies$ true in one recursive call;

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```
(define (euclid x y)
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Theorem: (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 2: Will show "$y \geq x/2$" $\implies$ "$mod(x, y) \leq x/2$."

Proof.

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mod (x, y) is second argument in next recursive call,

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First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 2: Will show “$y \geq x/2$” $\implies$ “$mod(x,y) \leq x/2$.”

mod (x,y) is second argument in next recursive call,
and becomes the first argument in the next one.

Proof.

```
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Proof:

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Case 2: Will show "$y \geq x/2$" $\Longrightarrow$ "$mod(x,y) \leq x/2$."
When $y \geq x/2$, then

$$\lfloor \frac{x}{y} \rfloor = 1,$$

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□

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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Computes the gcd(x, y) in $O(n)$ divisions.

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Computes the $\gcd(x, y)$ in $O(n)$ divisions.

For x and m, if $\gcd(x, m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

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What is multiplicative inverse of x modulo m?

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$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

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$$(3)12 + (-1)35 = 1.$$

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So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

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"Make d out of sum of multiples of x and y."

What is multiplicative inverse of x modulo m?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$\begin{aligned} ax + bm &= 1 \\ ax \equiv 1 - bm &\equiv 1 \pmod{m}. \end{aligned}$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
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Get 11 from 35 and 12 and plugin.... Simplify.

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

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```

Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.

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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example:

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ext-gcd(35,12)
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Extended GCD Algorithm.

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ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1$

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Example: $a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

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Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b$ $1 = \lfloor 35/12 \rfloor \cdot (-1) = 3$

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Theorem: Returns (d, a, b), where $d = gcd(a, b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

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Ind hyp: **ext-gcd**$(y, \mod(x, y))$ returns (d, a, b) with
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And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!

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Returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$.

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?

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Internet Security: Next Week!

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7(0) + 60(1) &= 60 \\
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Confirm:

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Confirm: $-119 + 120 = 1$

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$d = e^{-1} = -17 = 43 = \pmod{60}$