A: Is it possible to start and end at Sather Gate, such that you visit each Oski exactly once?

- possible
- impossible
1. Graphs

**Def** An (undirected simple) graph $G = (V, E)$ is defined by

- a set of vertices $V$ and a set of edges $E$, where elements in $E$ are of form \( \{u, v\} \) where $u, v \in V$, $u \neq v$.

**Eq. 1**

\[
\begin{align*}
V & = \{A, B, C, D\}, \\
E & = \{\{A, B\}, \{B, D\}, \{A, C\}\}.
\end{align*}
\]

- No multiple edges
- Not a simple graph, b/c.
- Not a simple graph, b/c, \( \{A, A\} \) not a set.

**Rem.** To model a directed graph $G = (V, E)$, we can define

\[ E \subseteq V \times V. \]

**Def** Given an edge $e = \{u, v\}$, we say $u \xrightarrow{e} v$

- $e$ is **incident** on vertices $u$ and $v$;
- $u$ and $v$ are **neighbors** or **adjacent**.

The degree of a vertex $u$ is $|\{v \in V : \{u, v\} \in E\}|$. 

**Thm** (The handshaking theorem) Let $G = (V, E)$ be a graph with $m$ edges. Then $2m = \sum_{v \in V} \deg(v)$.

**Pf:** Let $N$ be the number of pairs $(v, e)$ such that $v$ is an endpoint of $e$.

Since each $v$ belongs to $\deg(v)$ pairs, $\sum_{v \in V} \deg(v) = N$.

On the other hand, each edge belongs to 2 pairs, so $N = 2m$.

Hence, $2m = \sum_{v \in V} \deg(v)$. $\blacksquare$

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### 1.1 Eulerian Tours

**Def.** A **walk** is a sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_n, v_{n+1}\}$.

A **tour** is a walk that has no repeated edges, starts and ends at the same vertex.

A **Eulerian tour** is a tour that visits each edge exactly once.

**Rem.** A walk can be specified by a sequence of vertices in the order of visit.

**E.9.0** An Eulerian tour in $\begin{array}{c} 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1 \end{array}$ is $1, 2, 3, 4, 5, 3, 1$. 

\[ \begin{array}{c} 1 \end{array} \]
A graph is connected if there exists a path between any distinct \( u, v \in V \).

A connected graph \( G \) has an Eulerian tour iff every vertex has even degree.

\( \text{Pf: } 1 \) ("\( \Rightarrow \)") Assume \( G \) has an Eulerian tour starting at \( v_0 \).

For all \( v \in V \), pair up the two edges each time we enter and exit.

For \( v_0 \), additionally pair up the starting edge, and the ending edge.

Eulerian tour visits all edges exactly once,

\( \Rightarrow \forall v \in V \), incident edges are paired

\( \Rightarrow \forall v \in V \), \( \deg(v) \) is even.

\( 2 \) ("\( \Leftarrow \)") Assume every vertex in \( G \) has even degree.

Goal: Find an Eulerian tour.

Step 1: Pick an arbitrary \( v_0 \in V \) to start.

Keep following unvisited edges until stuck.

\( S, E, C, S, M, S \) is not an Eulerian tour because \( \{ M, E \} \) is not visited.
All degrees even $\Rightarrow$ stuck at $V_0$.

**Step 2:** Remove this tour.

Recurse on connected components.

**Step 3:** Splice the recursive tours into the main one to get a Eulerian tour.

E.g. Use the algorithm above to find an Eulerian tour in the following graph.
2. Special Graphs

2.1 Complete Graphs

A complete graph with \( n \) vertices, denoted \( K_n \), is a graph that contains every possible edge.

E.g. \( K_5 \)

2.2 Bipartite Graphs

A bipartite graph partitions vertices \( V \) into two disjoint sets \( V_1 \) and \( V_2 \) such that \( E \subseteq \{ \{ u, v \} : u \in V_1, v \in V_2 \} \).

A complete bipartite graph has \( E = \{ \{ u, v \} : u \in V_1, v \in V_2 \} \), denoted \( K_{m,n} \).

E.g. \( K_{3,3} \)
2.3. Hypercubes

An $n$-dim hypercube, denoted $Q_n$, has a vertex for each length-$n$ bit string, and an edge between a pair of vertices iff they differ in one bit.

Rem. Hypercubes can be constructed recursively. To build $Q_{n+1}$ from $Q_n$,

- make two copies of $Q_n$,
- prefacing 0 for one copy and 1 for the other,
- add edges between copies of corresponding vertices.

E.g. $Q_1$, $Q_2$, $Q_3$

2.4. Trees

Def. A cycle is a tour, s.t. the only repeated vertex is the start and end vertex.

Def. A tree is a connected, acyclic graph.

A leaf is a vertex of degree 1.
Rem. Try to prove leaf lemmas:

"every tree has at least one leaf" and
"a tree minus a leaf remains a tree".

They allow us to do induction on trees!!!

**Thm**

T is a tree connected, no cycle.

\[ \iff \quad T = (V, E) \text{ is connected and has } |V| - 1 \text{ edges} \]

**Pf:**

1. ("\(\Rightarrow\)"") We'll do induction on \(n = |V|\), i.e.,

   \[ P(n) : \text{tree } T \text{ has } n \text{ vertices } \Rightarrow T \text{ has } n-1 \text{ edges}. \]

   **Base case:** \(n=1. \quad n-1=0. \quad \checkmark \)

   \[ P(n-1) \Rightarrow P(n). \]

   **Inductive Step:** Suppose \(T\) has \(n\) vertices.

   - By leaf lemmas, we can remove a leaf & its incident edge to get
     a tree \(T'\) with \(n-1\) vertices.
     
   - By IH, \(T'\) has \((n-1)-1 = n-2\) edges.

     \[ \Rightarrow T \text{ has } n-2+1 = n-1 \text{ edges}. \]

2. ("\(\Leftarrow\)"") We'll do induction on \(n=|V|\).

   \[ P(n) : T \text{ is connected, has } n-1 \text{ edges } \Rightarrow T \text{ is a tree} \]

   **Base case:** \(n=1. \quad \checkmark \)

   \[ P(n-1) \Rightarrow P(n). \]

   **Inductive Step:** Suppose \(T\) connected, has \(n-1\) edges.

   - By handshaking theorem, total degree \(= 2(n-1) = 2n - 2\).

     \[ \Rightarrow \exists v \in V, \deg(v) = 1. \text{ Remove a vertex of degree } 1 \text{ and its incident edge.} \]

     \[ \Rightarrow \text{ add } v \text{ and its edge, we still get a connected graph, and creates no cycles. } \Rightarrow T \text{ is a tree.} \]
**Def.** A cycle is a tour where the only repeated vertices are the start and end vertices.

**Thm.** The following statements are all equivalent:

- $T$ is connected and contains no cycle.
- $T$ is connected and has $|V| - 1$ edges.
- $T$ is connected, and removing any edge disconnects $T$.
- $T$ has no cycle, and adding any single edge creates a cycle.