1) Combinatorial Theorem

2) Simple Inclusion and Exclusion

3) Inclusion and Exclusion

4) Derangements

5) Sampling

6) Star and Bars

\[ 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \]

LHS: \[ 2 \times 2 \times \cdots \times 2 = 2^n \] The number of subsets of \{1, \ldots, n\}

RHS:

\[
\begin{array}{c|c}
\text{subset size} & 0 \text{ or } 1 \text{ or } 2 \cdots \text{ or } n \\
\hline
\binom{n}{0} & \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \\
2^n & = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}
\end{array}
\]
1) Combinatorial Theorem:

\[(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n} y^n\]

How to distribute \(n\) balls among \(x\) red bins and \(y\) blue bins?

LHS: \((x+y)(x+y)(x+\cdots\cdot(x+y))\)

Make subsequent choices:

\(x+y\)
\(x+y\)
\(x+y\)
\(\cdots\)
\(x+y\)

\[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (x+y)^n\] possibilities.

RHS: or \[i = 0 \rightarrow n\]

\(n\) balls in red
\(0\) ball in blue

\[\binom{n}{0} x^n + \binom{n}{1} x^{n-1} \cdots + \binom{n}{n} y^n\]

\[(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i\]

\(x=1, y=1 \Rightarrow 2^n = \sum_{i=0}^{n} \binom{n}{i}\)

\(x=1, y=-1 \Rightarrow -2^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i}\)
2) Simple Inclusion/Exclusion:

Sum Rule: For disjoint sets A and B ($|A \cap B| = 0$) to count the number of elements of $A \cup B$, we have $|A \cup B| = |A| + |B|$

when A and B have common element:

Inclusion-Exclusion Rule:

$|A \cup B| = |A| + |B| - |A \cap B|$

Elements in $|A \cap B|$ are counted twice $\Rightarrow$ subtract $|A \cap B|$
Example: How many 10-digit phone numbers have 5 as their first or second digit?

\[
A = \text{numbers with 5 as first digit, } |A| = 10^9 \\
B = \text{numbers with 5 as second digit, } |B| = 10^9 \\
A \cup B = 10^9 \\
A \cap B = 10^8 \\
|A \cup B| = |A| + |B| - |A \cap B| = 10^9 + 10^9 - 10^8.
\]

- Three way inclusion-exclusion rule:

Set \( A, B, C \)

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\]
3) Inclusion-Exclusion Principle:

sets $A_1, \ldots, A_n$

$$|U_i A_i| = \sum_i |A_i| - \sum_{i_1 \neq i_2} |A_{i_1} \cap A_{i_2}| + \cdots + (-1)^{n-1} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} |A_{i_1} \cap \cdots \cap A_{i_n}|.$$ 

4) Derangements:

Permutations of $1, \ldots, n$? $n!$

How many permutations where no item in its proper place or fixed points (Derangements)?

Example: number of derangements $123$?

Derangements?

$\left\{ \begin{array}{c}
123 \text{ no!} \\
213 \text{ no!} \\
231 \text{ yes}
\end{array} \right.$
We can count the complement: count permutations with at least one fixed point.

\[ A_i = \text{"Permutation where } i \text{ is fixed point"} \]

\[ \sum_{i=1}^{3} |A_i| = 1A_1 + 1A_2 + 1A_3 - 1A_1 \cap A_2 \]

\[ = 2! + 2! + 2! - 1 - 1 - 1 + 1 \]

\[ = 4 \]

Subtract this from the total permutations.

For \( n \) items: \# Derangements = \( n! - 4 \) = 2

Permutations with at least one fixed point?

\[ \sum_{i=1}^{n} |A_i| = \sum_{i} 1A_i - \sum_{i \neq j} 1A_i \cap A_j + \cdots + (-1)^{n-1} \sum_{i \neq j \neq \cdots \neq n} 1A_i \cap \cdots \cap A_n \cdot \]

\[ \left( \binom{n}{1} (n-1)! \right) + \left( \binom{n}{2} (n-2)! \right) \cdots + (-1)^{n-1} \left( \binom{n}{n} 0! \right) \]
\[ \text{# of derangements} = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \ldots + (-1)^n \frac{n!}{n} \]

\[ = n! - \frac{n!}{(n-1)!} + \frac{n!}{(n-2)!} + \ldots + \frac{n!}{n!}(-1)^n \]

\[ = n! \times \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} \right) \]

\[ = n! \times \sum_{i=0}^{n} \frac{(-1)^i}{i!} \]

\[ n! \times \frac{1}{e} = 0.37n! \quad \text{as} \quad n \to \infty \]

Roughly 0.37 of the permutations are derangements!

5) Sampling:

Assume sample \( k \) items out of \( n \):

\[
\begin{aligned}
\text{Without replacement} & \\
\text{Order matters:} & \quad n \times (n-1) \times (n-2) \ldots \times (n-k+1) \quad = \frac{n!}{(n-k)!} \\
\text{Order does not matter:} & \\
\text{Second rule: divide by number of orders} \quad & = \frac{n!}{(n-k)! \cdot k!}
\end{aligned}
\]
• With replacement:

• order matters: \( n \times n \times \ldots \times n = n^k \)

• order matters: can we use second rule? doesn't

Problem? depends on how many of each item we choose.

For chosen string \( ABCD \rightarrow 4! \) orderings

\[ \ldots A A A A C D \rightarrow \frac{4!}{2!} \] orderings

• Different number of ordered elements map to each unordered.

Another example:

How many ways can Alice and Bob split $5?

For each of 5 dollars pick Alice or Bob

\( 2^5 \) and divide out order
A: 5 B: 0 : (A, A, A, A, A) * 1
A: 3 B: 2 : (A, A, A, B, B), ... * 10
A: 2 B: 3 : (A, A, B, B, B), ... * 10
A: 1 B: 4 : (A, B, B, B, B), ... * 5
A: 0 B: 5 : (B, B, B, B, B) * 1

**Second rule of counting** is no good here.

Another example!

How many ways can Alice, Bob, and Eve split $5?*

Idea: separate Alice's dollars from Bob's and then Bob's from Eve's.

Assume dollars are 5 stars: * * * *

Split: Alice: 2, Bob: 1, Eve: 2

Split: Alice: 0, Bob: 2, Eve: 3

\[ \leftrightarrow \text{ Stars and bars } * * * * * \]
Zeroth Rule Counting: If there is a one-to-one mapping between two set they have the same size.

So we can ask: How many different sequence of 5 stars and 2 bars are there?

\[ \begin{array}{cccccccc}
\wedge & \wedge & \wedge & \wedge & \wedge & \wedge & \wedge & \wedge \\
\wedge & \wedge & \wedge & \wedge & \wedge & \wedge & \wedge & \wedge \\
\end{array} \]

7 Positions in which to Place 2 bars.

\[ \binom{7}{2} \] ways to split \$5 among 3 people.

6) Star and Bars:

\( k \) ways to split \( k \) dollars among \( n \) people.

\[ \binom{k+n-1}{n-1} \]

Correspondence: \( n-1 \) bars to split the \( k \) stars.
\( n+k-1 \) positions from which to choose

\( n-1 \) bar positions:

\[ \binom{n+k-1}{n-1} \].