Joint, Conditional, and Marginal PDFs

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Joint PDFs

Let $X$ and $Y$ be two continuous random variables. Then the joint density function $f_{X,Y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y) \, dy \, dx$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy \, dx = 1$$

and

$$f_{X,Y}(x,y) \geq 0 \quad \forall x, y \in \mathbb{R}$$
Joint CDFs

Let \( X \) and \( Y \) be two random variables. Then the joint cumulative distribution function \( F_{X,Y} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies:

\[
F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y)
\]

In the continuous case, assuming the CDF is continuous:

\[
f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)
\]

\[
\frac{\partial^2}{\partial y \partial x} \text{ is also fine.}
\]
Conditional PDFs

\[ \text{Just like } \ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

Let \( X \) and \( Y \) be two continuous random variables with joint density function \( f_{X,Y} \).
For any \( y \) with \( f_y(y) > 0 \), the conditional distribution of \( X \) given \( Y = y \) is defined as:

\[ f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_y(y)} \]

When \( Y \) is continuous, even though \( P(Y = y) = 0 \), if \( f_y(y) > 0 \), then:

\[ P(a \leq X \leq b \mid Y = y) = \int_a^b f_{X \mid Y}(x \mid y) \, dx \]
Let $X$ and $Y$ be two continuous random variables. $X$ and $Y$ are independent if:

$$f_{x,y}(x, y) = f_x(x)f_y(y)$$

for all $x, y$. Since $f_{x,y}(x, y) = f_{x|y}(x|y)f_y(y)$, this implies $f_{x|y}(x|y) = f_x(x)$.  

\[\text{Just like } P(A \cap B) = P(A|B)P(B) \text{ in discrete setting}\]

\[\text{in order for } x \text{ and } Y \text{ to be independent}\]
Marginalization

To recover the individual pdfs from the joint pdf:

\[ f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \]

\[ f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \]

For \( f_X(x) \) to be a valid pdf, \( \int_{-\infty}^{\infty} f_X(x) \, dx = 1 \)

\( f_X(x) \) is a marginal pdf of \( f_{X,Y}(x, y) \)
Example: $X, Y \overset{i.i.d.}{\sim} \text{Unif}(0, 2)$

What is $f_{X,Y}(x,y)$ for two uniform rvs on $[0, 2]$?

Uniform on 2x2 square $\Rightarrow$ Constant

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy \, dx = 1
\]

\[
\int_{0}^{2} \int_{0}^{2} c \, dy \, dx = 1
\]

\[
c \int_{0}^{2} \int_{0}^{2} 1 \cdot dy \, dx = 1
\]

\[
c \cdot 4 = 1
\]

\[
c = \frac{1}{4}
\]

\[
f_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & \text{if } (x,y) \in A \\ 0 & \text{else} \end{cases}
\]
Example: $X, Y \overset{i.i.d.}{\sim} \text{Unif}(0, 2)$

What is $f_{X,Y}(x,y)$ for two uniform rvs on $[0, 2]$? We know that it must be a nonzero constant $c$ on the two-by-two square, since all $x,y$ pairs are equally likely. Again, we can use the constraint:

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy \, dx = 1
\quad \Rightarrow \quad
\int_{0}^{2} \int_{0}^{2} c \, dy \, dx = 1
\quad \Rightarrow \quad
4 \cdot c = 1
\quad \Rightarrow \quad
c = \frac{1}{4}
$$
Uniform Density Over a Disk: Joint

What is $f_{X,Y}(x,y)$ for a uniform density over a disk of radius $r$ centered at the origin?

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } x^2 + y^2 \leq r^2 \\ 0 & \text{o.w.} \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy = \int_{x^2+y^2 \leq r^2} c \, dx \, dy$$

$$= c \int_{x^2+y^2 \leq r^2} 1 \, dx \, dy$$

$$= c \cdot \text{(area of disk)}$$

$$= c \cdot \pi r^2$$

We know pdf must integrate to 1, so $c \cdot \pi r^2 = 1 \Rightarrow c = \frac{1}{\pi r^2}$
Uniform Density Over a Disk: Joint

What is \( f_{X,Y}(x, y) \) for a uniform density over a disk of radius \( r \) centered at the origin?

\[
f(x, y) = \begin{cases} 
    c, & \text{if } x^2 + y^2 \leq r^2 \\
    0, & \text{otherwise}
\end{cases}
\]

\[
\int \int_{x^2+y^2\leq r^2} c \cdot dx \cdot dy = c \cdot \int \int_{x^2+y^2\leq r^2} 1 \cdot dx \cdot dy
\]

\[
= c \cdot \text{[area of disk]}
\]

\[
= c \cdot \pi \cdot r^2
\]

By definition, the joint must integrate to 1, so

\[
1 = c \cdot \pi \cdot r^2 \Rightarrow c = \frac{1}{\pi \cdot r^2}
\]
Uniform Density Over a Disk: Marginals

What is $f_y(y)$, and $f_x(x)$, now that we know the joint over the disk?

\[
f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y) \, dx
\]

\[
= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2} \, dx
\]

\[
= \frac{1}{\pi r^2} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} 1 \, dx
\]

\[
= 2 \cdot \frac{\sqrt{r^2-y^2}}{\pi r^2} \quad \text{(for } -r \leq y \leq r)\]

\[
f_x(x) = 2 \cdot \frac{\sqrt{r^2-x^2}}{\pi r^2} \quad \text{(for } -r \leq x \leq r)\]

\[
= 0 \quad \text{o.w.}
\]
Uniform Density Over a Disk: Marginals

What is $f_y(y)$, and $f_x(x)$, now that we know the joint over the disk?

\[
f_y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi \cdot r^2} \cdot dx\]

\[
= \frac{1}{\pi \cdot r^2} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \cdot dx
\]

\[
= \frac{2\sqrt{r^2 - y^2}}{\pi \cdot r^2}
\]

for $-r \leq y \leq r$

By symmetry,

\[
f_x(x) = \frac{2\sqrt{r^2 - y^2}}{\pi \cdot r^2}
\]
What is \( f_{x|y}(x|y) \)?

By definition, \( f_{x|y}(x|y) = \frac{f_{x,y}(x,y)}{f_y(y)} \)

\[
= \frac{1}{\pi r^2} \cdot \frac{1}{2\sqrt{r^2 - y^2}}
\]

\( \Rightarrow X \text{ and } Y \text{ are not independent} \)

What about \( \text{Cov}(X,Y) \)?

\[
\begin{align*}
\text{E}[x] &= 0 \\
\text{E}[y] &= 0 \\
\text{E}[x|y=y] &= 0
\end{align*}
\]

by symmetry

\[
E[x|y=y] = y \cdot \text{E}[x|y=y]
\]

\[
= y \cdot 0 = 0
\]

\( \Rightarrow E[xy] = 0 \)

\[
\text{Cov}(X,Y) = E[XY] - E[X]E[Y]
\]

\( = 0 - 0 = 0 \)

\( X,Y \text{ are uncorrelated, but are dependent} \)
Uniform Density Over a Disk: Conditional PDFs

What is \( f_{x|y}(x|y) \)? If \( x^2 + y^2 \leq r^2 \) and \( |y| < r \),

\[
f_{x|y}(x|y) = \frac{f_{x,y}(x,y)}{f_y(y)} = \frac{1}{\pi r^2} \cdot \frac{2\sqrt{r^2-y^2}}{2\sqrt{r^2-y^2}} = \frac{1}{2 \sqrt{r^2-y^2}}
\]

Since this differs from \( f_x(x) \), we know \( X \) and \( Y \) are dependent. Furthermore,

\[
\mathbb{E}[X] = \mathbb{E}[Y] = 0 \text{ by symmetry}
\]

\[
\mathbb{E}[X|Y = y] = 0 \text{ by symmetry}
\]

\[
\mathbb{E}[XY|Y = y] = y \cdot \mathbb{E}[X|Y = y] = y \cdot 0 = 0 \Rightarrow \mathbb{E}[XY] = 0
\]

\[
\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0
\]

\( X, Y \) are uncorrelated but dependent!!
Let $X, Y$ be two random variables with joint PDF $f_{X,Y}(x, y)$, and let $g(x, y)$ be a real-valued function of $x, y$. Then

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) \, dx \, dy$$

You don't need to find the joint pdf of $g(X, Y)$ in order to calculate $\mathbb{E}[g(X, Y)]$. Use the known joint pdf of $X$ and $Y$. 
Expected Distance Between Two Points

Let \( X, Y \overset{i.i.d.}{\sim} \text{Unif}(0, 1) \). What is \( \mathbb{E}[|X - Y|] \)?

\[
\mathbb{E}[|X - Y|] = \int_0^1 \int_0^1 |x - y| f_{X,Y}(x,y) \, dx \, dy
\]

\[
= \int_0^1 \int_0^y (y - x) \, dx \, dy + \int_0^1 \int_y^1 (x - y) \, dx \, dy
\]

\[
= 2 \int_0^1 \int_y^1 (x - y) \, dx \, dy
\]

\[
= 2 \int_0^1 \left( \frac{x^2}{2} - xy \right) \bigg|_{x=y} \, dy
\]

\[
= 2 \int_0^1 \left( \frac{y^2}{2} - y^2 + \frac{y}{2} \right) \, dy
\]

\[
= \frac{1}{3}
\]
Expected Distance Between Two Points

Let \( X, Y \overset{i.i.d.}{\sim} Unif(0, 1) \). What is \( \mathbb{E}[|X - Y|] \)?

\[
\mathbb{E}[|X - Y|] = \int_0^1 \int_0^1 |x - y| f_{X,Y}(x,y) \, dx \, dy
\]

(1)

\[
= \int_0^1 \int_0^1 |x - y| \cdot 1 \, dx \, dy
\]

(2)

\[
= \int \int_{X > Y} (x - y) \, dx \, dy + \int \int_{Y > X} (y - x) \, dx \, dy
\]

(3)

\[
= 2 \cdot \int_0^1 \int_y^1 (x - y) \, dx \, dy
\]

(4)

\[
= 2 \cdot \int_0^1 \left( \frac{x^2}{2} - xy \right) \bigg|_{x=y}^1 \, dy
\]

(5)

\[
= \frac{1}{3}
\]

(6)

(7)
### Total Probability Theorem

**Discrete Partition**

- \[ P(A \cap B) = \sum_{i=1}^{n} P(B_i) \cdot P(A \mid B_i) \]
- \[ f_x(x) = \sum_{i=1}^{n} P(B_i) \cdot f_{x \mid B_i}(x) \]

**Continuous Partition**

- \[ P(A) = \int_{-\infty}^{\infty} f_x(x) \cdot P(A \mid X = x) \, dx \]
- \[ f_x(x) = \int_{-\infty}^{\infty} f_y(y) \cdot f_{x \mid y}(x \mid y) \, dy \]

**Notes**

- \( f_x(x) \to \text{continuous} \)
- \( P(x = x) \to \text{discrete} \)
Total Probability Theorem Examples

Discrete/Continuous:
\[ P(Y > X) = \int_{-\infty}^{\infty} f_X(x) \cdot P(Y > X \mid X = x) \, dx \]

Continuous/Discrete:
Flip a fair coin. If its heads, then \( X \sim \text{Exp}(\lambda_1) \), otherwise \( X \sim \text{Exp}(\lambda_2) \). Then, for \( x > 0 \),

\[ f_X(x) = \frac{1}{2} \cdot \lambda_1 e^{-\lambda_1 x} + \frac{1}{2} \cdot \lambda_2 e^{-\lambda_2 x} \]
Bayes’ Rule

| Dis. | \( P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{\sum_{j=1}^{n} P(A_j) \cdot P(B|A_j)} \) | Cont. | \( P(A_i|X = x) = \frac{P(A_i) \cdot f_{X|A_i}(x)}{\int_{-\infty}^{\infty} f_X(t) P(A|X = t) dt} \) |
|------|---------------------------------------------------------------------------------|-------|---------------------------------------------------------------------------------|
| Dis. | \( f_{X|A}(x) = \frac{f_X(x) \cdot P(A|X = x)}{\int_{-\infty}^{\infty} f_X(t) P(A|X = t) dt} \) | Cont. | \( f_{X|Y}(x|y) = \frac{f_X(x) \cdot f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t) f_{Y|X}(Y|t) dt} \) |

Discrete/Continuous: \[ \text{for right} \]

\[
P(A|X = x) = P(A|X \in [x, x + \delta]) = \frac{P(A) \cdot P(X \in [x, x + \delta]|A)}{P(A) \cdot P(X \in [x, x + \delta]|A) + P(A^c) \cdot P(X \in [x, x + \delta]|A^c)}
\]

\[
= \frac{P(A) \cdot f_{X|A}(x) \cdot \delta}{P(A) \cdot f_{X|A}(x) \cdot \delta + P(A^c) \cdot f_{X|A^c}(x) \cdot \delta}
\]

\[
= \frac{P(A) \cdot f_{X|A}(x)}{P(A) \cdot f_{X|A}(x) + P(A^c) \cdot f_{X|A^c}(x)}
\]