Invariant Distribution Recap

A distribution $\pi$ is *invariant* for the transition probability matrix $P$ if it satisfies the following *balance equations*:

$$\pi = \pi P$$

(1)
1. A state $i$ is recurrent if starting from $i$, no matter what path we take, we can always return to $i$

2. A state $i$ is transient if starting from $i$, there exists a path for which there is no way back to $i$

3. A class of states is a set of states where it is possible to get from any state to any other state
Irreducibility Definition

A Markov chain is irreducible if it can go from every state $i$ to every other state $j$, possibly in multiple steps.
Irreducibility Example

irreducible
bcz you can go from any state i to any state j

reducible
bcz you cannot go from 1 to 2
for example.

reducible
bcz you cannot go from 2 to 0
for example.
For an irreducible Markov Chain, we have that:

1. The chain has a unique invariant distribution \( \pi = [\pi(1) \ldots \pi(n)] \).
2. For each \( j \in X \),

\[
\lim_{n \to \infty} \frac{\sum_{m=0}^{n-1} \mathbf{1}\{X_m = j\}}{n} = \pi(j)
\]

This holds regardless of what particular \( \pi_0 \) we use.
Consider an irreducible Markov chain on \( \mathcal{X} \) with transition probability matrix \( P \). Define

\[
d(i) := \text{g.c.d}\{n > 0 \mid P^n(i, i) = \Pr[X_n = i \mid X_0 = i] > 0\}, \ i \in \mathcal{X}.
\]

1. Then, \( d(i) \) has the same value for all \( i \in \mathcal{X} \). If that value is 1, the Markov chain is said to be aperiodic. Otherwise, it is said to be periodic with period \( d \).

2. If the Markov chain is aperiodic, then

\[
\Pr[X_n = i] \to \pi(i), \ \forall i \in \mathcal{X}, \ \text{as} \ n \to \infty. \tag{2}
\]

where \( \pi \) is the unique invariant distribution.

For a given state \( i \), the quantity \( d(i) \) is the greatest common divisor of all the integers \( n > 0 \) so that the Markov chain can go from state \( i \) to state \( i \) in \( n \) steps.
Periodicity Example

Path lengths from 1 to 1:
\{1, 2, 3, \ldots\}

\(d(1) = 1\)

Path lengths from 0 to 0:
\{3, 6, 9, \ldots\}

\(d(0) = 3\)

Periodic:

Path lengths from 1 to 0:
\{2, 3, \ldots\}

\(d(1) = 1\)

Path lengths from 0 to 0:
\{3, 6, 9, \ldots\}

Periodically not defined since it is not irreducible.
Key Points

1. If a Markov chain is irreducible, it has a unique stationary distribution but does not necessarily converge to it.

2. Periodicity is not defined for reducible Markov chains.

3. If a Markov chain contains a self-loop, it is aperiodic. If there isn’t a self-loop, it may or may not be aperiodic.

4. If a Markov chain is **irreducible and aperiodic**, then it converges to a unique invariant distribution regardless of the initial distribution $\pi_0$.

\[
\pi_0 \quad \xrightarrow{P} \quad \pi_0 \, P \quad \xrightarrow{\pi_0 \, P^2} \quad \pi_0 \, P^3 \rightarrow \pi
\]