

Probability and CS70

Lecture 16, CS70 Fall 2026

Introduction

- **Probability and CS70**
- Probability Spaces
- Event Probability

Event Probability Examples

- Counting Sample Points
- Counting Weighted Sample Points
- Poker Hands
- Balls and Bins
- The Birthday Paradox
- Summary of Examples

A 9 Percent Chance

"A team of researchers from the University of Hawai'i at Mānoa [published a study this week](#) that estimated the probability of a magnitude 9+ earthquake in the Aleutian Islands—an event with sufficient power to create a mega-tsunami especially threatening to Hawai'i. In the next 50 years, they report, there is a 9 percent chance of such an event." - [The School of Ocean and Earth Science and Technology at the University of Hawai'i at Mānoa](#)

Probability and Intuition

Suppose we flip a coin 20 times. Which outcome is most likely?

- HHHHHHHHHHHHHHHHHHHH
- HHTTHTHHHTTHTHTHTH

<https://joshh.ug/A>



Suppose we flip a coin 20 times. Which outcome is most likely?

- HHHHHHHHHHHHHHHHHHHH
- HHTTHTHHHTTHTHTHTH

They are equally likely. Both occur with probability $1/2^{20}$.

Probability and Intuition

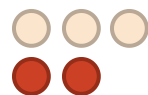
Suppose we have two shopping centers.



Shopping center 1:

3 Indian restaurants

2 Moroccan restaurants



Shopping center 2:

1 Indian restaurant

1 Moroccan restaurant



Suppose we pick a center randomly with probability 50% and then pick a restaurant at that center with equal probability.

- If we eat at a Moroccan restaurant, what is the chance we picked center 1?

In this second half of CS70, we'll give you a solid foundation for probability.

- What did the "9%" mean in that chance for a mega-tsunami?
- Which outcome is more likely for coin flipping?
- What is the chance we picked shopping center 1?

Probabilistic reasoning is famously counterintuitive!

- A firm foundation will keep you safe from easy mistakes.

This second phase of the class is a bit of a restart.

We'll use what we learned about proofs (e.g. direct proofs, contrapositive, induction, etc).

Other topics like graphs, modular arithmetic, error correcting codes, countability, computability, stable matching, and so forth will be unused.

Counting Problems Summary

- yOnR: Ordered Sampling without Replacement, a.k.a. counting sets of sequences.
- yOyR: Ordered Sampling with Replacement, a.k.a. counting sets of sequences.
- nOnR: Unordered Sampling without Replacement, a.k.a. counting sets of sets.
- nOyR: Unordered Sampling with Replacement, a.k.a. counting sets of multisets.

The number of possible samples if we draw k items from n possibilities is:

- $|\text{yOnR}(n, k)| = \frac{n!}{(n-k)!} = n \times (n-1) \times \cdots \times (n-k+1)$

- $|\text{yOyR}(n, k)| = n^k$

- $|\text{nOnR}(n, k)| = \binom{n}{k} = \frac{|\text{yOnR}(n, k)|}{k!}$

- $|\text{nOyR}(n, k)| = |\text{nOnR}(n+k-1, n-1)| = \binom{n+k-1}{n-1}$ If we count the positions of the separators
 $= |\text{nOnR}(n+k-1, k)| = \binom{n+k-1}{k}$ If we count the positions of the non-separators

$$\binom{N}{K} = \binom{N}{N-K}$$

Probability Spaces

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Random Experiment, Sample Point, Sample Space

Random Experiment: Drawing a sample of k items from a set S with cardinality n .

Example: Rolling two 4-sided dice. In this case $k = 2$, $S = \{1, 2, 3, 4\}$, and $n = 4$, and we are drawing with replacement.



Sample Point: The outcome of a single experiment, e.g. 22.

Sample Space: Set of all possible outcomes, given by Ω . In this case, the sample space is exactly one of our 4 counting problems, i.e. this is yOyR for $n = 4$, $k = 2$.

$$\Omega = \begin{array}{cccc} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{array}$$

Probability Space

A **Probability Space** consists of:

- A sample space Ω
- A probability function $\mathbf{P}(\omega)$ that maps $\omega \in \Omega$ to a real number such that:
 - $0 \leq \mathbf{P}(\omega) \leq 1$ for all $\omega \in \Omega$
 - $\sum_{\omega \in \Omega} \mathbf{P}(\omega) = 1$



Example: Rolling two fair 4-sided dice. In this case, Ω and $\mathbf{P}$ are:

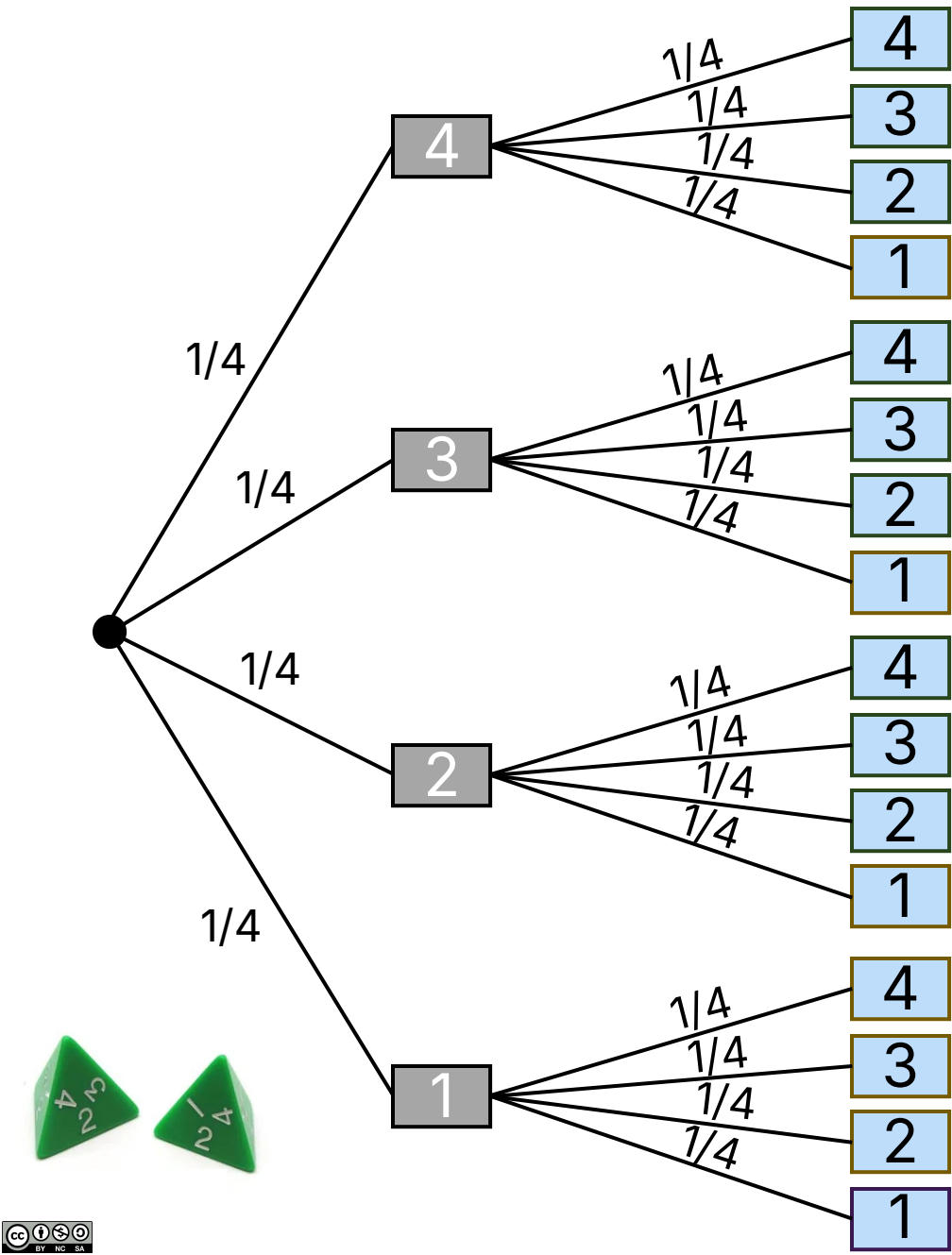
$$\Omega = \begin{array}{cccc} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{array}$$

$$\mathbf{P}(\omega) = \frac{1}{16}$$

This is a “uniform probability space”.

Tree Diagram for a Uniform Probability Space

$$P(\omega) = \frac{1}{16}$$



Can also think about die rolls as a tree of possibilities.

- First level is the first roll.
- Second level is the second roll.

Example Non-Uniform Probability Space

Example: Rolling two unfair 4-sided dice. In this case $S = \{1, 2, 3, 4\}$, $k = 2$, and we are drawing with replacement. Suppose that dice are loaded so that 1 appears $5/8$ of the time, and other numbers occur $1/8$ of the time.

In this example $\mathbf{P}(\omega)$ is slightly more complicated:

- $\mathbf{P}(\mathbf{11}) = ??$



<https://joshh.ug/B>



$\Omega =$

11	12	13	14
21	22	23	24
31	32	33	34
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In this example $\mathbf{P}(\omega)$ is slightly more complicated:

- $\mathbf{P}(\mathbf{11}) = 25/64$



$\Omega =$

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

This is a non-uniform probability space.

Example Non-Uniform Probability Space

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In this example $P(\omega)$ is slightly more complicated:

- $P(11) = 25/64$
- $P(21) = P(31) = P(41) = P(12) = P(13) = P(14) =$



$\Omega =$

11	12	13	14
21	22	23	24
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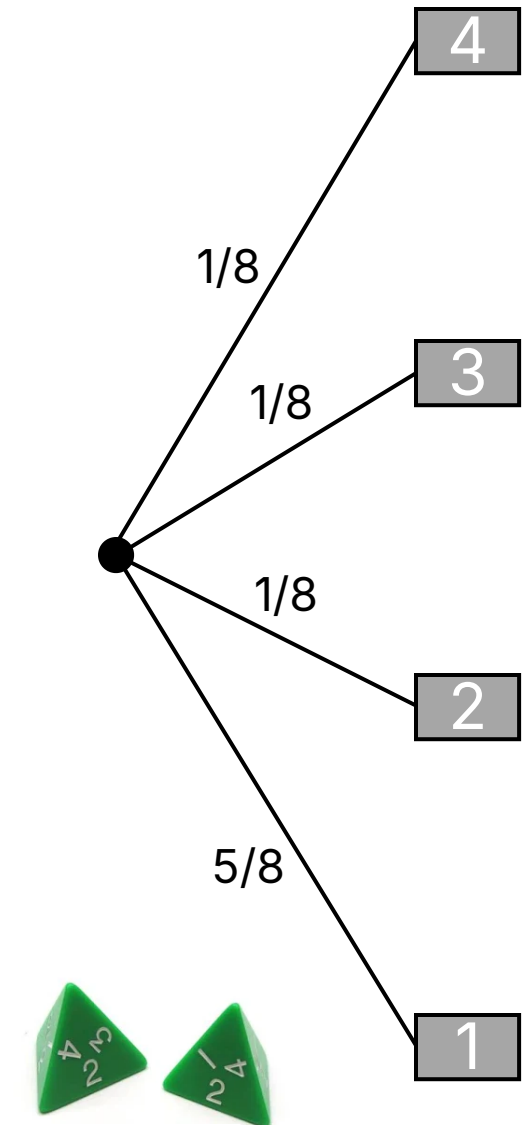
- $P(11) = 25/64$
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- $P(22) = P(23) = \dots = P(44) = 1/64$



$$\Omega = \begin{array}{cccc} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{array}$$

This is a non-uniform probability space.

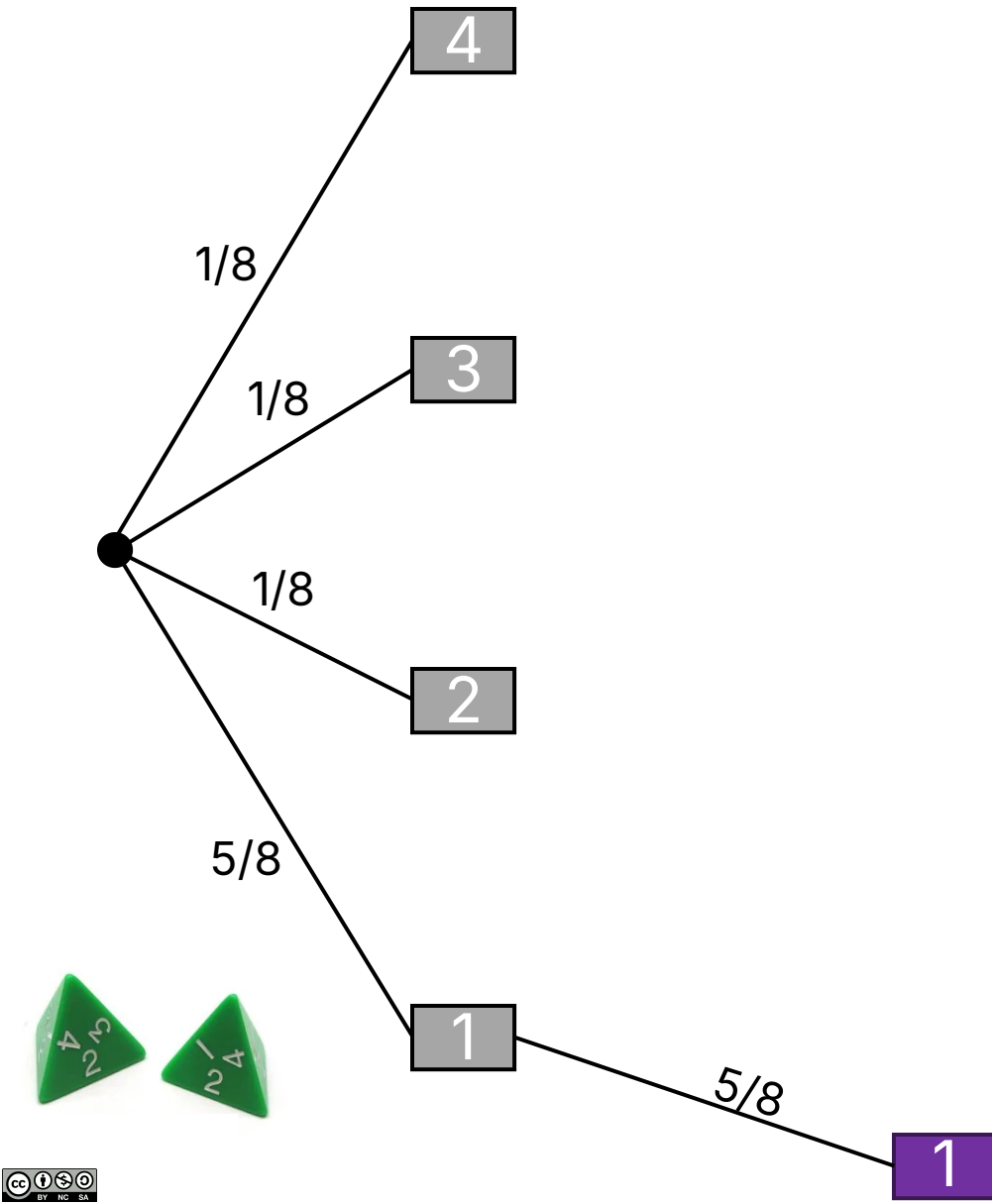
Tree Diagram for a Non-Uniform Probability Space



A tree diagram gives a somewhat more formal explanation of the probability of each outcome.

Tree Diagram for a Non-Uniform Probability Space

- $P(\text{11}) = \frac{5}{8} \times \frac{5}{8} = 25/64$

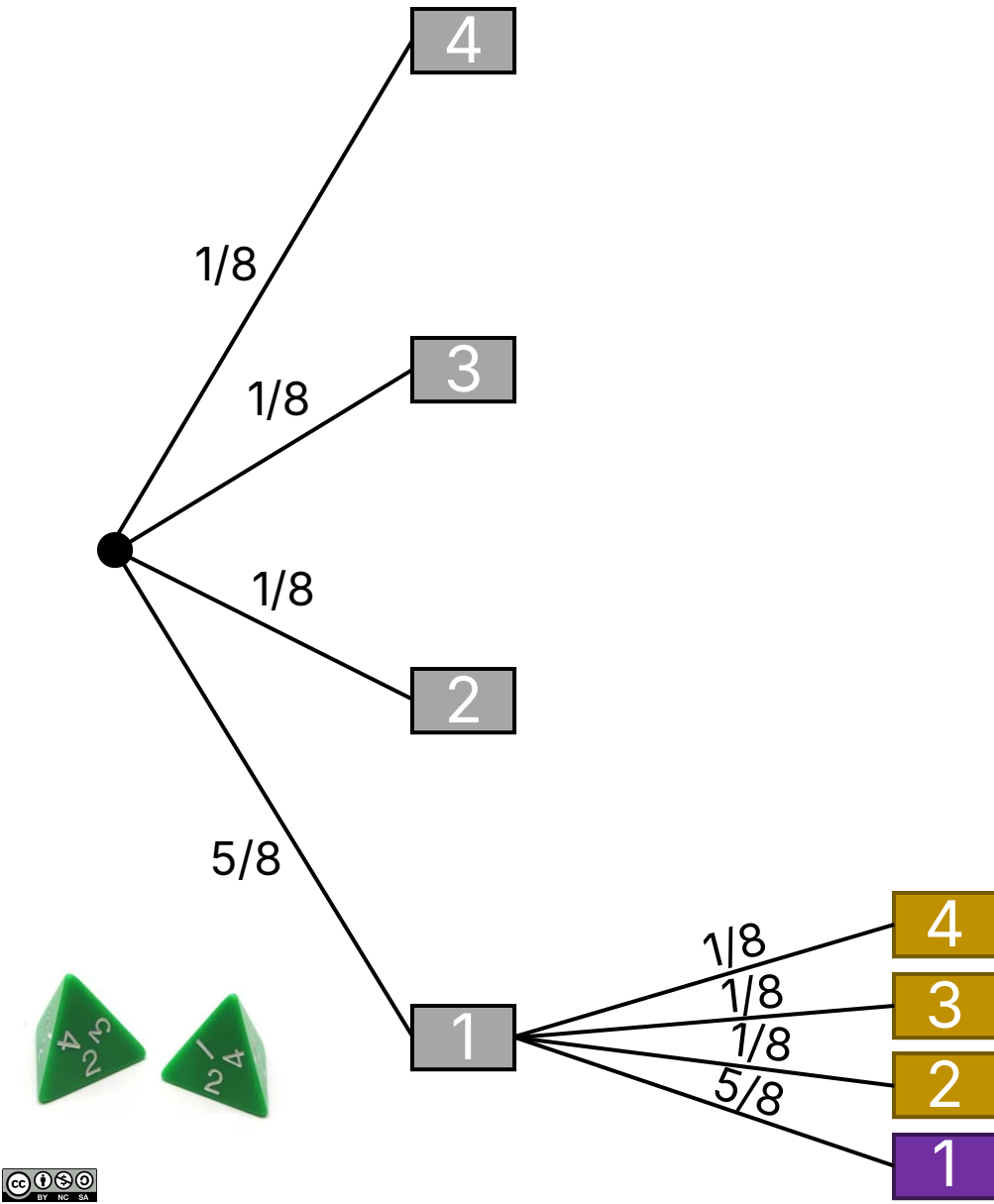


A tree diagram gives a somewhat more formal explanation of the probability of each outcome.

We can multiply because the dice rolls are **independent**: the first die shows 1 on 5/8 of rolls, and on 5/8 of those the second die shows 1 too. So 11 comes up $5/8 \times 5/8 = 25/64$ of the time. More on independence in Lecture 18.

Tree Diagram for a Non-Uniform Probability Space

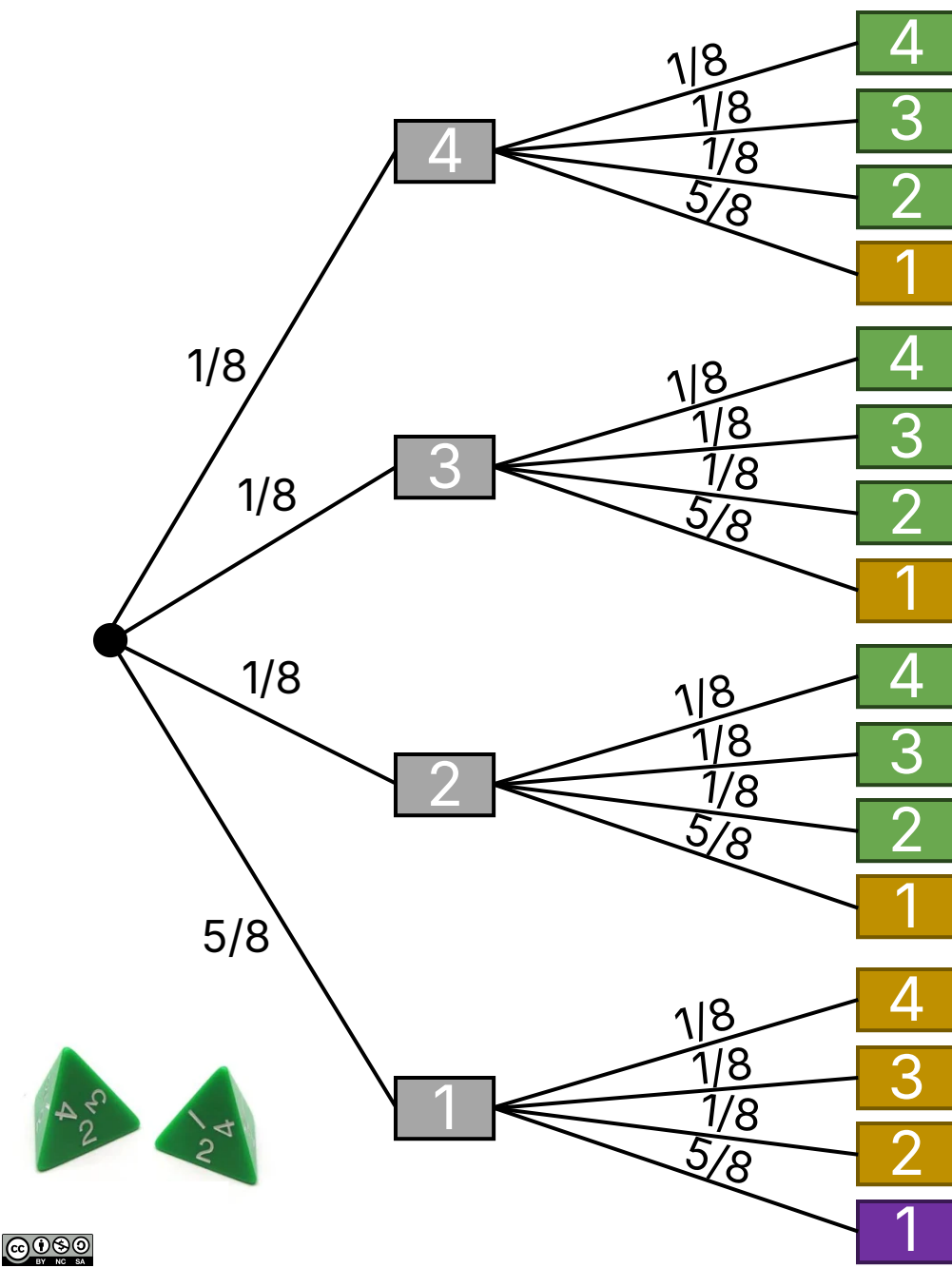
- $P(\textcolor{violet}{11}) = \frac{5}{8} \times \frac{5}{8} = 25/64$
- $P(\textcolor{brown}{12}) = P(\textcolor{brown}{13}) = P(\textcolor{brown}{14}) = \frac{5}{8} \times \frac{1}{8} = 5/64$



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We can multiply because the dice rolls are **independent**: the first die shows 1 on 5/8 of rolls, and on 5/8 of those the second die shows 1 too. So 11 comes up $5/8 \times 5/8 = 25/64$ of the time. More on independence in Lecture 18.

Tree Diagram for a Non-Uniform Probability Space



- $P(11) = 25/64$
- $P(12) = P(13) = P(14) = P(21) = P(31) = P(41) = 5/64$
- $P(22) = P(23) = \dots = P(44) = 1/64$

A tree diagram gives a somewhat more formal explanation of the probability of each outcome.

We can multiply because the dice rolls are **independent**: the first die shows 1 on $5/8$ of rolls, and on $5/8$ of those the second die shows 1 too. So 11 comes up $5/8 \times 5/8 = 25/64$ of the time. More on independence in Lecture 18.

Reflection on Notation

My slides use parentheses for probability functions, e.g.

$$\mathbf{P}(\omega)$$

The notes use brackets and a blackboard P:

$$\mathbb{P}[\omega]$$

Reminder: Mathematical symbols are just placeholders for ideas.

- These two notations are exactly equivalent.
- Other notations exist.

Reflection on Problem Framing: yOyR vs. nOyR

We thought of our set in terms of yOyR (ordered w/replacement).

- Also valid to define in terms of nOyR (unordered w/replacement).

nOyR view: For **two four sided dice**: $\Omega = \{11, 12, 13, 14, 22, 23, 24, 33, 34, 44\}$

- 10 outcomes rather than 16 because $|\text{nOyR}(4, 2)| = \binom{5}{3} = \binom{5}{2} = 10$.

E.g. as binary strings with 1s as separators: 10101 = 23, 11100 = 44.

- Bad idea! nOyR yields a **non-uniform** probability space even with fair dice. Harder to work with. For this reason, we'll usually prefer yOyR over nOyR.

- $P(11) = 1/16$

- $P(12) = 1/8$

$$\Omega_{\text{yOyR}} = \begin{array}{cccc} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{array}$$

$$\Omega_{\text{nOyR}} = \begin{array}{cccc} 11 & 12 & 13 & 14 \\ & 22 & 23 & 24 \\ & & 33 & 34 \\ & & & 44 \end{array}$$

Reflection on Problem Framing: yOyR vs. nOyR

To get uniform probability spaces, we'll usually prefer yOyR over nOyR.

- For fair dice:
 - All outcomes $\omega \in \Omega_{\text{yOyR}}$ have the same chance of occurring.
 - Outcomes $\omega \in \Omega_{\text{nOyR}}$ have different chances of occurring.
- Note: We don't ACTUALLY care about the order of the dice, but pretending we do makes calculations easier.

yOyR Framing

$\Omega_{\text{yOyR}} =$

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

nOyR Framing

$\Omega_{\text{nOyR}} =$

11	12	13	14
	22	23	24
		33	34
			44

Event Probability

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Event

An **event** is a subset of our sample space Ω . Often given by the letter A , maybe with some subscripts. Examples:

- A_{F1} : Our first roll is a 1.

$$A_{F1} = \{11, 12, 13, 14\}$$

$\Omega =$

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

- $A_{SumNotPrime}$: Sum of dice is not prime.

$$A_{SumNotPrime} = \{13, 22, 31, 24, 33, 42, 44\}$$

$\Omega =$

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44



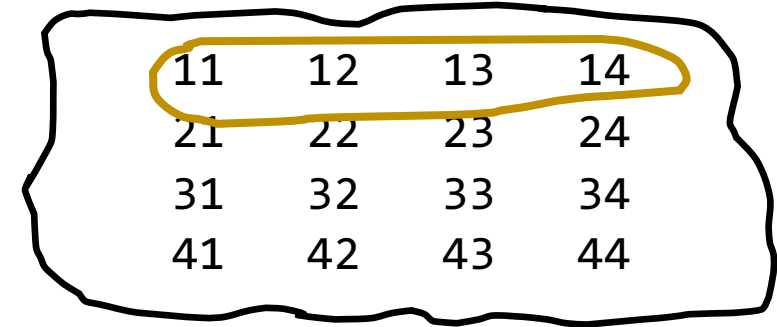
Event

The probability of **event** is the sum of the probability of the individual samples in the event, i.e. $P(A) = \sum_{\omega \in A} P(\omega)$

- A_{F1} : Our first roll is a 1.

$$A_{F1} = \{11, 12, 13, 14\}$$

$\Omega =$



11	12	13	14
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$$P(A_{F1}) = P(11) + P(12) + P(13) + P(14)$$

For fair dice:

- $P(A_{F1}) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} = 25\%$

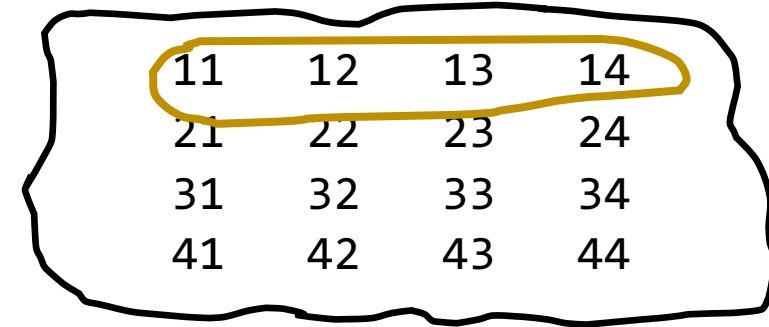
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- A_{F1} : Our first roll is a 1.

$$A_{F1} = \{11, 12, 13, 14\}$$

$\Omega =$



11	12	13	14
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$$\mathbf{P}(A_{F1}) = \mathbf{P}(11) + \mathbf{P}(12) + \mathbf{P}(13) + \mathbf{P}(14)$$

For loaded dice where 1 appears 5/8 of the time:

- $\mathbf{P}(A_{F1}) = \frac{25}{64} + \frac{5}{64} + \frac{5}{64} + \frac{5}{64} = \frac{40}{64} = 62.5\%$

From earlier:

- $\mathbf{P}(11) = 25/64$
- $\mathbf{P}(12) = \mathbf{P}(13) = \mathbf{P}(14) = 5/64$

Understanding Check

Suppose we roll two loaded 4 sided dice where 1 appears 5/8 of the time, i.e.

- $P(\textcolor{violet}{11}) = 25/64$
- $P(\textcolor{brown}{21}) = P(\textcolor{brown}{31}) = P(\textcolor{brown}{41}) = P(\textcolor{brown}{12}) = P(\textcolor{brown}{13}) = P(\textcolor{brown}{14}) = 5/64$
- $P(\textcolor{green}{22}) = P(\textcolor{green}{23}) = \dots = P(\textcolor{green}{44}) = 1/64$



What is $P(A_{2\text{nd} \text{greater}})$?

$\Omega =$

$\textcolor{violet}{11}$	$\textcolor{brown}{12}$	$\textcolor{brown}{13}$	$\textcolor{brown}{14}$
$\textcolor{brown}{21}$	$\textcolor{green}{22}$	$\textcolor{green}{23}$	$\textcolor{green}{24}$
$\textcolor{brown}{31}$	$\textcolor{green}{32}$	$\textcolor{green}{33}$	$\textcolor{green}{34}$
$\textcolor{brown}{41}$	$\textcolor{green}{42}$	$\textcolor{green}{43}$	$\textcolor{green}{44}$

<https://joshh.ug/C>



Understanding Check

Suppose we roll two loaded 4 sided dice where 1 appears 5/8 of the time, i.e.

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What is $P(A_{2ndgreater})$?

$\Omega =$

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

$$\begin{aligned} P(A_{2ndgreater}) &= 3 \times 5/64 + 3 \times 1/64 \\ &= 18/64 \\ &= 9/32 \end{aligned}$$

Counting Sample Points

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Another Example: Flipping Coins

Suppose we flip a fair coin 16 times.

- What is the sample space Ω ? Its size?
- What is the probability of any specific outcome?



Another Example: Flipping Coins

Suppose we flip a fair coin 16 times.

- What is the sample space Ω ? Its size?
 - This is just yOyR again with $n = 2$, $k = 16$.
 - $\Omega = \{\text{HHHHHHHHHHHHHHHHHH}, \text{HHHHHHHHHHHHHHHHHT}, \dots\}$
 - $|\Omega| = 2^{16}$
- What is the probability of any specific outcome?
 - $P(\text{HHHHHHHHHHHHHHHHHH}) = P(\text{HHHHHHHHHHHHHHHHHT}) = \dots = \frac{1}{2^{16}}$



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 - $\Omega = \{\text{HHHHHHHHHHHHHHHHHH}, \text{HHHHHHHHHHHHHHHHHT}, \dots\}$
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- What is the probability of any specific outcome?
 - $P(\text{HHHHHHHHHHHHHHHHHH}) = P(\text{HHHHHHHHHHHHHHHHHT}) = \dots = \frac{1}{2^{16}}$
- Let A_{6H} be the event where we get exactly six heads. What is the probability $P(A_{6H})$?



Another Example: Flipping Coins

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 - $\Omega = \{\text{HHHHHHHHHHHHHHHHHH}, \text{HHHHHHHHHHHHHHHHHT}, \dots\}$
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- What is the probability of any specific outcome?
 - $P(\text{HHHHHHHHHHHHHHHHHH}) = P(\text{HHHHHHHHHHHHHHHHHT}) = \dots = \frac{1}{2^{16}}$
- Let A_{6H} be the event where we get exactly six heads. What is the probability $P(A_{6H})$?
 - $P(A_{6H}) = P(\text{HHHHHHTTTTTTTTTT}) + P(\text{THHHHHHTTTTTTTTTT}) + \dots$
 - How many samples have six heads?



Another Example: Flipping Coins

Suppose we flip a fair coin 16 times.

- What is the sample space Ω ? Its size?
 - This is just yOyR again with $n = 2$, $k = 16$.
 - $\Omega = \{\text{HHHHHHHHHHHHHHHHHH}, \text{HHHHHHHHHHHHHHHHHT}, \dots\}$
 - $|\Omega| = 2^{16}$
- What is the probability of any specific outcome?
 - $P(\text{HHHHHHHHHHHHHHHHHH}) = P(\text{HHHHHHHHHHHHHHHHHT}) = \dots = \frac{1}{2^{16}}$
- Let A_{6H} be the event where we get exactly six heads. What is the probability $P(A_{6H})$?
 - $P(A_{6H}) = P(\text{HHHHHHHTTTTTTTTTTT}) + P(\text{THHHHHHTTTTTTTTTTT}) + \dots$
 - How many samples have six heads? This is given by nOnR with $n = 16$, $k = 6$.
 - $|A_{6H}| = \binom{16}{6} = 8008$
 - So $P(A_{6H}) = \binom{16}{6} / 2^{16} \approx 12.2\%$



$$\text{Or } \binom{16}{10} = 8008$$

Sixteen Coins Summary

Entire sample space given by yOyR with $n = 2$, $k = 16$.

- That is $\Omega = \{\text{HHHHHHHHHHHHHHHHHH}, \text{HHHHHHHHHHHHHHHHHT}, \dots\}$
- $|\Omega| = 2^{16}$



Probability of a certain number of heads is computed via nOnR.

- Set of outcomes with 6 heads: $A_{6H} = \{\text{HHHHHHTTTTTTTTTT}, \dots\}$
- A_{6H} is given by nOnR with $n = 16$, $k = 6$.
 - $|A_{6H}|$ is the number of ways to get 6 heads out of 16, i.e. $|A_{6H}| = \binom{16}{6} = 8008$
 - We are "choosing" the position of our 6 heads.

Can also think of $P(A_{6H})$ as the fraction of the outcomes that belong to A_{6H} , i.e.

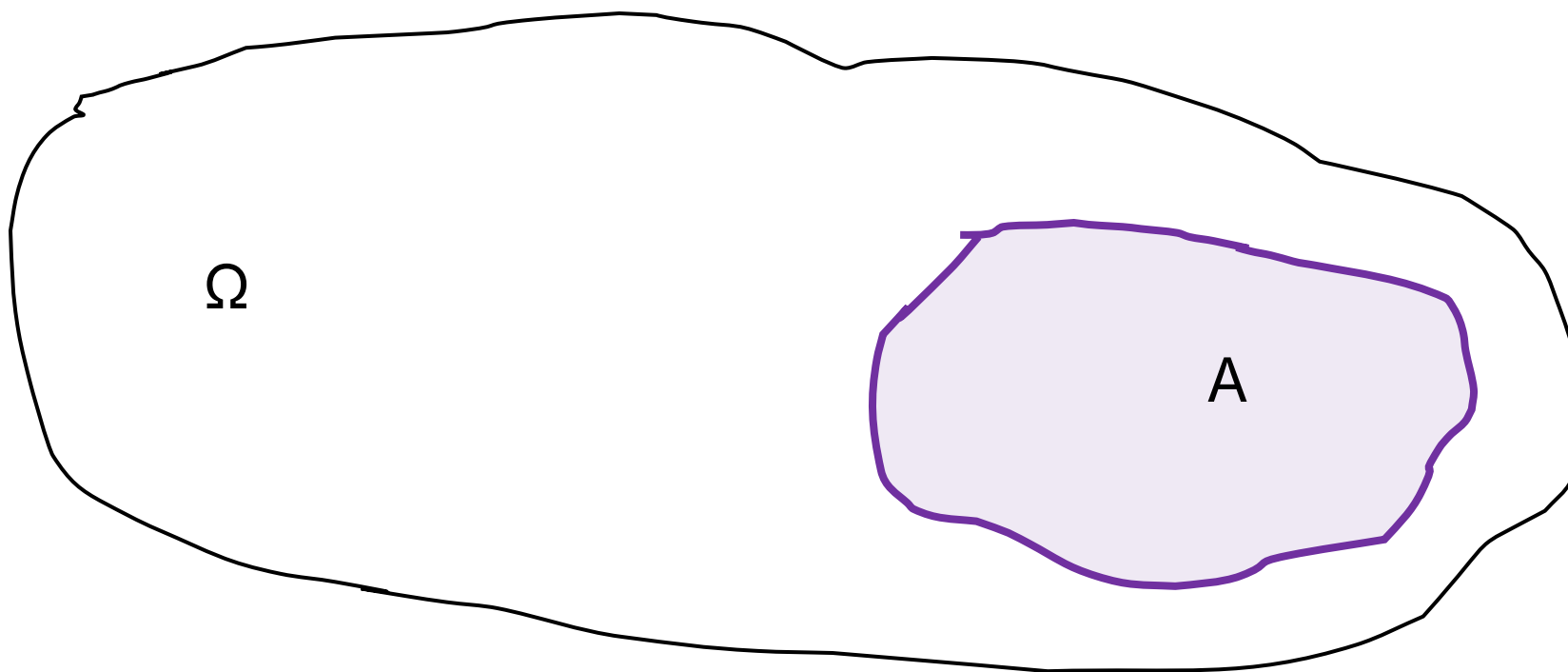
$$P(A_{6H}) = \frac{|A_{6H}|}{|\Omega|} = 8008/2^{16} \approx 12.2\%$$

A vs. Ω

In a uniform probability space, the probability of an event A is the fraction of the sample space that belongs to A .

- That is $\sum_{\omega \in A} \mathbf{P}(\omega) = |A|/|\Omega|$ for a uniform probability space.

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|}$$

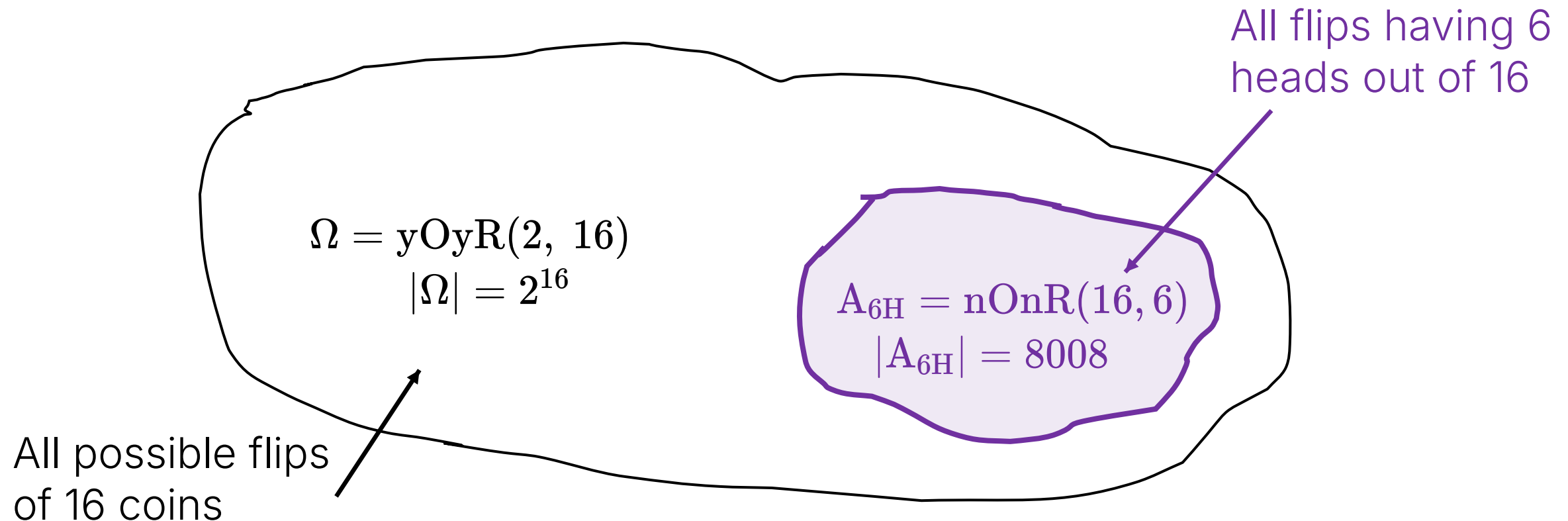


Probability Calculations

Common technique for solving probability problems.

- Simply compute the relative size of two sets.
- Very often, our set sizes in CS70 will be given by one of our four counting problems.

$$P(A_{6H}) = \frac{|A_{6H}|}{|\Omega|}$$



Counting Weighted Sample Points

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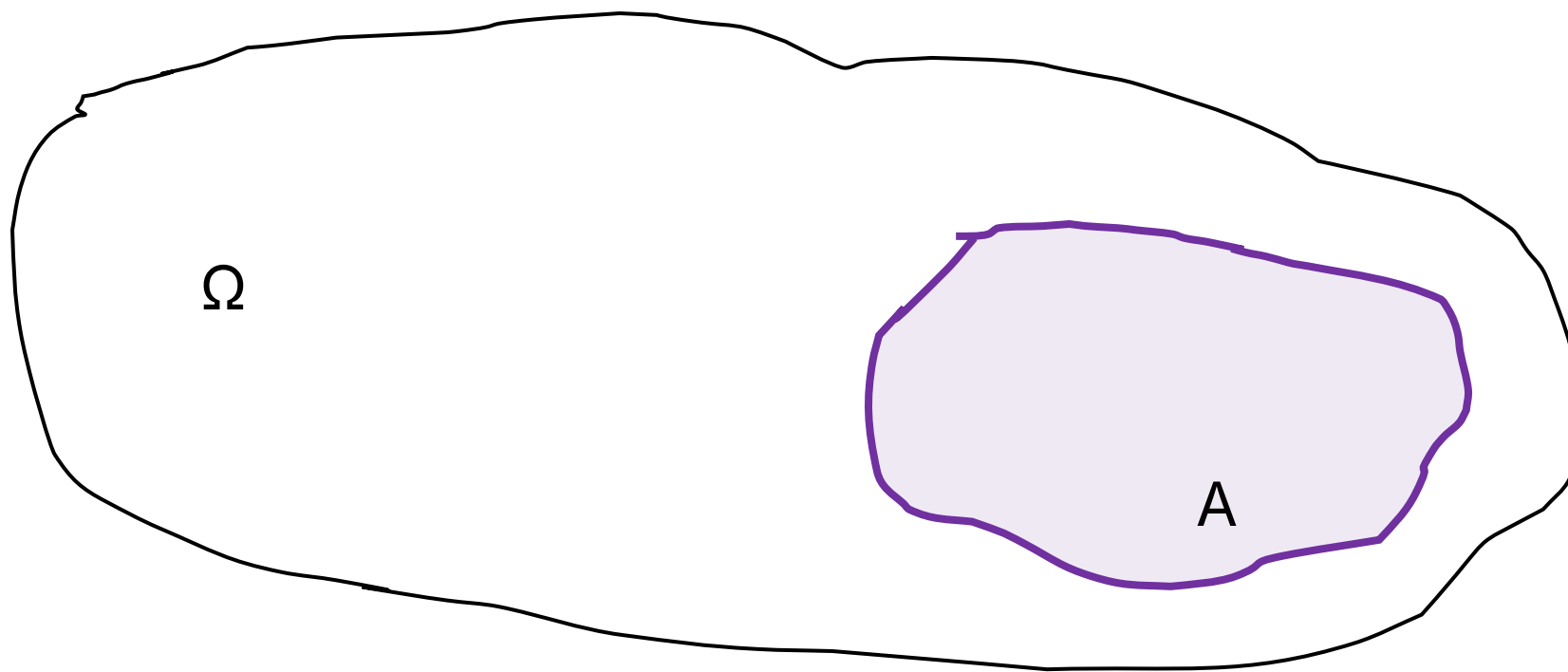
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A vs. Ω

For a uniform probability space, the probability of an event is given by the ratio of the number of sample points in A to the number of sample points in Ω

$$P(A) = \frac{|A|}{|\Omega|}$$



For non-uniform probability spaces, formula doesn't work. Must weight samples.

Four Biased Coins

One common model in engineering applications is the biased coin. Can be used to model, e.g. chance that a system will fail.

Suppose we have a coin where $P(H) = 1/3$ and $P(T) = 2/3$.



Suppose that our experiment consists of four consecutive flips.

What is the chance that we get HHHH? The chance that we get HTHT?

$\Omega =$

HHHH	HHHT	HHTH	HHTT
HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT
TTHH	TTHT	TTTH	TTTT

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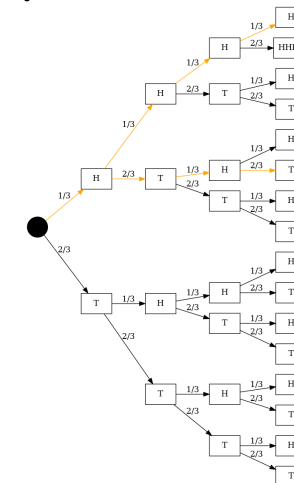
Suppose that our experiment consists of four consecutive flips.

What is the chance that we get HHHH? The chance that we get HTHT?

- $P(HHHH) = (1/3)^4$, $P(HTHT) = (1/3)^2(2/3)^2$

$\Omega =$

HHHH	HHHT	HHTH	HHTT
HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT
TTHH	TTHT	TTTH	TTTT



Non-uniform!

Four Biased Coins

One common model in engineering applications is the biased coin. Can be used to model, e.g. chance that a system will fail.



Suppose we have a coin where $\mathbf{P}(H) = 1/3$ and $\mathbf{P}(T) = 2/3$.

Suppose that our experiment consists of four consecutive flips.

What is the chance $\mathbf{P}(A_{2H})$ that we have two heads?

$\Omega =$

HHHH	HHHT	HHTH	HHTT
HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT
TTHH	TTHT	TTTH	TTTT

<https://joshh.ug/D>



Four Biased Coins

Suppose we have a coin where $P(H) = 1/3$ and $P(T) = 2/3$.

Suppose that our experiment consists of four consecutive flips.



What is the chance $P(A_{2H})$ that we have two heads?

- First, compute probability of any specific two heads outcome, e.g. $P(HHTT)$
 - $P(HHTT) = P(HTHT) = \dots = P(TTHH) = (2/3)^2(1/3)^2 = 4/81$
- Then add the six probabilities: $P(HHTT) + P(HTHT) + \dots + P(TTHH)$
 - $P(A_{2H}) = 6 \times 4/81 = 24/81$

$\Omega =$

HHHH	HHHT	HHTH	<u>HHTT</u>
HTHH	<u>HTHT</u>	<u>HTTH</u>	HTTT
THHH	<u>THHT</u>	<u>THTH</u>	THTT
<u>TTHH</u>	TTHT	TTTH	TTTT

Note: $\binom{4}{2} = 6$

Sixteen Biased Coins

Suppose we have a coin where $\mathbf{P}(H) = 1/3$ and $\mathbf{P}(T) = 2/3$.

Suppose that our experiment consists of 16 consecutive flips.

- What is the chance $\mathbf{P}(A_{6H})$ that we have exactly 6 heads?



Sixteen Biased Coins

Suppose we have a coin where $\mathbf{P}(H) = 1/3$ and $\mathbf{P}(T) = 2/3$.

Suppose that our experiment consists of 16 consecutive flips.

- What is the chance $\mathbf{P}(A_{6H})$ that we have exactly 6 heads?



Chance of any single specific outcome with 6 heads: $(1/3)^6(2/3)^{10}$

- $\mathbf{P}(HHHHHHHTTTTTTTTTT) = (1/3)^6(2/3)^{10}$
- $\mathbf{P}(THHHHHHTTTTTTTTTT) = (1/3)^6(2/3)^{10}$

Sixteen Biased Coins

Suppose we have a coin where $P(H) = 1/3$ and $P(T) = 2/3$.

Suppose that our experiment consists of 16 consecutive flips.

- What is the chance $P(A_{6H})$ that we have exactly 6 heads?



Chance of any single specific outcome with 6 heads: $z_6 = (1/3)^6(2/3)^{10}$

- $P(HHHHHHTTTTTTTTTT) = (1/3)^6(2/3)^{10} = z_6$
- $P(THHHHHHTTTTTTTTTT) = (1/3)^6(2/3)^{10} = z_6$
- ...

So $P(A_{6H})$ is?

<https://joshh.ug/E>



Sixteen Biased Coins

Suppose we have a coin where $\mathbf{P}(H) = 1/3$ and $\mathbf{P}(T) = 2/3$.

Suppose that our experiment consists of 16 consecutive flips.

- What is the chance $\mathbf{P}(A_{6H})$ that we have exactly 6 heads?



Chance of any single specific outcome with 6 heads: $z_6 = (1/3)^6(2/3)^{10}$

- $\mathbf{P}(HHHHHHHTTTTTTTTTTTT) = (1/3)^6(2/3)^{10} = z_6$
 - $\mathbf{P}(THHHHHHHHTTTTTTTTTTTT) = (1/3)^6(2/3)^{10} = z_6$
 - ...
- } Add these together, but how many?

So $\mathbf{P}(A_{6H})$ is? $\mathbf{P}(HHHHHHHTTTTTTTTTTTT) + \mathbf{P}(THHHHHHHHTTTTTTTTTTTT) + \dots$

Sixteen Biased Coins

Suppose we have a coin where $\mathbf{P}(H) = 1/3$ and $\mathbf{P}(T) = 2/3$.

Suppose that our experiment consists of 16 consecutive flips.

- What is the chance $\mathbf{P}(A_{6H})$ that we have exactly 6 heads?



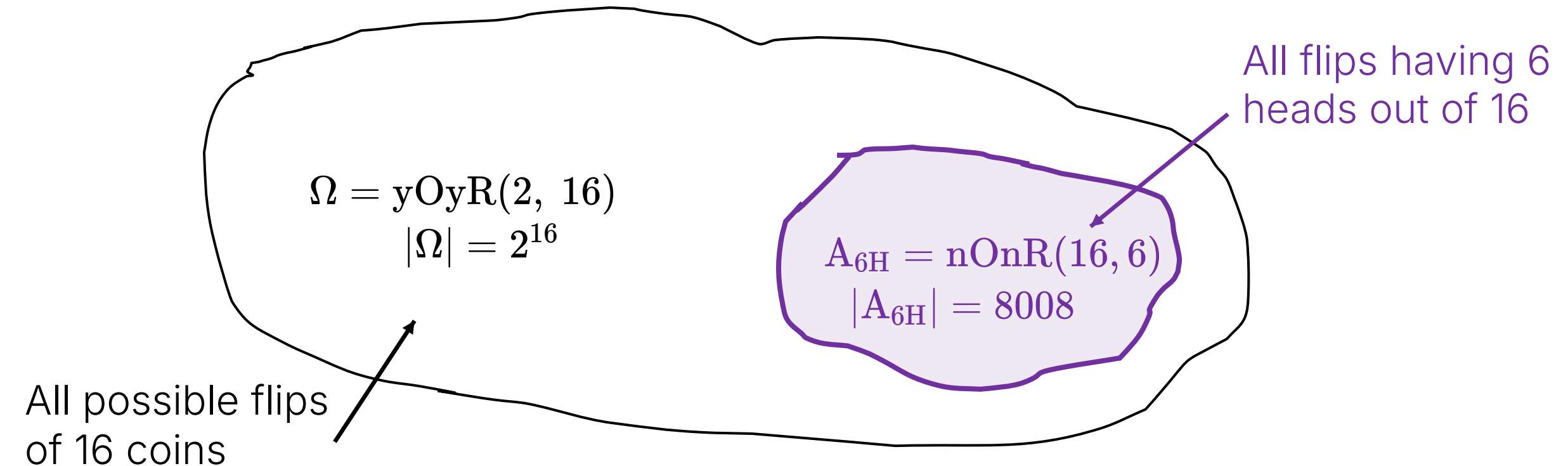
Chance of any single specific outcome with 6 heads: $z_6 = (1/3)^6(2/3)^{10}$

- $\mathbf{P}(HHHHHHHTTTTTTTTTT) = (1/3)^6(2/3)^{10} = z_6$
 - $\mathbf{P}(THHHHHHTTTTTTTTTT) = (1/3)^6(2/3)^{10} = z_6$
 - ...
- There are $\binom{16}{6}$ of these.
Sums to $\mathbf{P}(A_{6H}) = \binom{16}{6} z_6$

So $\mathbf{P}(A_{6H})$ is? $\binom{16}{6} (1/3)^6 (2/3)^{10} = \sim 19\%$

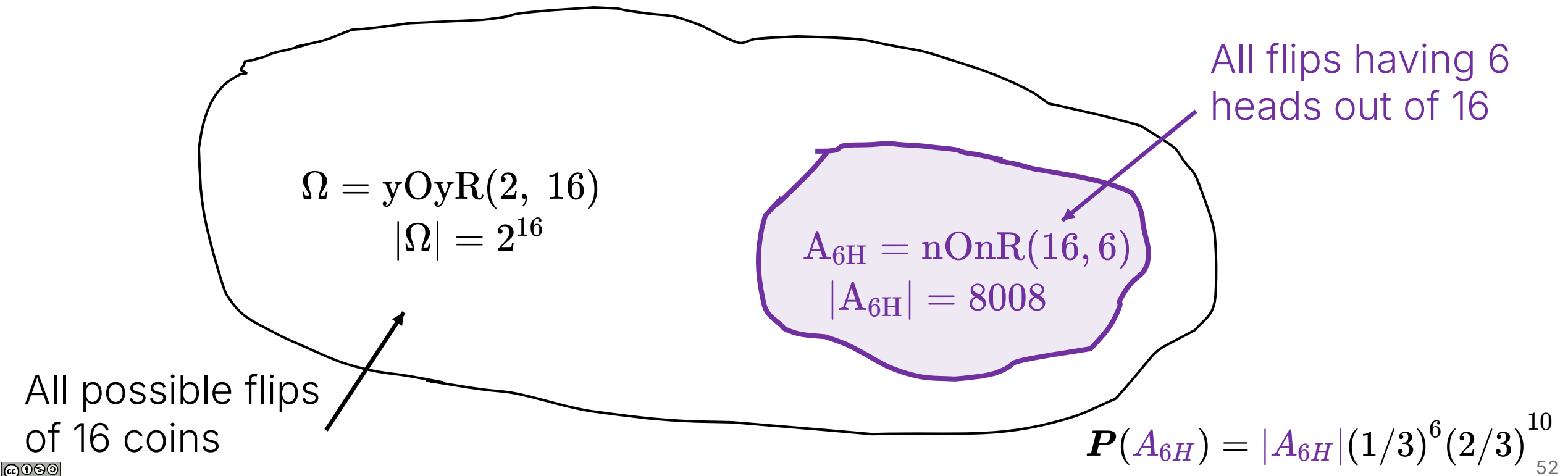
On a non-uniform probability space, probability is not based solely on relative set sizes.

$$P(A_{6H}) \neq \frac{|A_{6H}|}{|\Omega|} = \frac{8008}{2^{16}}$$



On a non-uniform probability space, we must weight outcomes appropriately.

- Instead of $\mathbf{P}(A) = \frac{|A|}{|\Omega|}$, we must use the general formula $\mathbf{P}(A) = \sum_{\omega \in A} \mathbf{P}(\omega)$.
- All outcomes in A_{6H} have probability $(1/3)^6(2/3)^{10}$, so $\mathbf{P}(A_{6H}) = |A_{6H}|(1/3)^6(2/3)^{10}$



Sixteen Biased Coins: A More Complicated Event

Suppose we have a coin where $P(H) = 1/3$ and $P(T) = 2/3$.

Suppose that our experiment consists of 16 consecutive flips.

- What is the chance $P(A_{<4H})$ that we have fewer than 4 heads?



Chance of any single specific outcome with 0 heads: $z_0 = (1/3)^0(2/3)^{16}$

- And there's only one of these.

Chance of any single specific outcome with 1 head: $z_1 = (1/3)^1(2/3)^{15}$

- And there are 16 of these.

Sixteen Biased Coins: A More Complicated Event

Suppose we have a coin where $P(H) = 1/3$ and $P(T) = 2/3$.

Suppose that our experiment consists of 16 consecutive flips.

- What is the chance $P(A_{<4H})$ that we have fewer than 4 heads?



i	Chance of any specific way to get i heads	Number of ways to get i heads
0	$(1/3)^0 (2/3)^{16}$	$\binom{16}{0} = 1$
1	$(1/3)^1 (2/3)^{15}$	$\binom{16}{1} = 16$
2	$(1/3)^2 (2/3)^{14}$	$\binom{16}{2} = 120$
3	$(1/3)^3 (2/3)^{13}$	$\binom{16}{3} = 560$

Sixteen Biased Coins: A More Complicated Event

Suppose we have a coin where $P(H) = 1/3$ and $P(T) = 2/3$.

Suppose that our experiment consists of 16 consecutive flips.

- What is the chance $P(A_{<4H})$ that we have fewer than 4 heads?

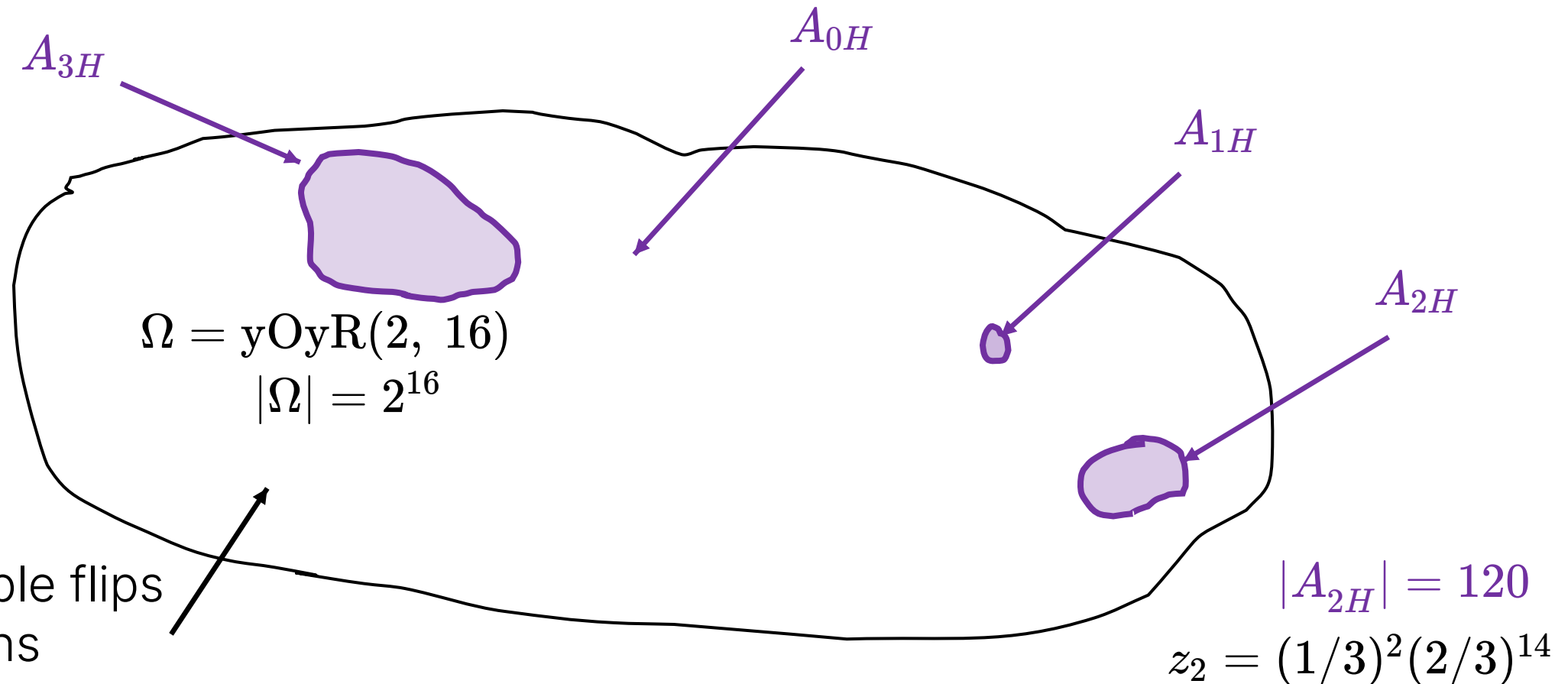


i	Chance of any specific way to get i heads	Number of ways to get i heads
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2	$(1/3)^2 (2/3)^{14}$	$\binom{16}{2} = 120$
3	$(1/3)^3 (2/3)^{13}$	$\binom{16}{3} = 560$

$$P(A_{<4H}) = (1/3)^0 (2/3)^{16} + 16 \times (1/3)^1 (2/3)^{15} + 120 \times (1/3)^2 (2/3)^{14} + 560 \times (1/3)^3 (2/3)^{13}$$

We have that $\mathbf{P}(A_{<4H}) = \mathbf{P}(A_{0H}) + \mathbf{P}(A_{1H}) + \mathbf{P}(A_{2H}) + \mathbf{P}(A_{3H})$.

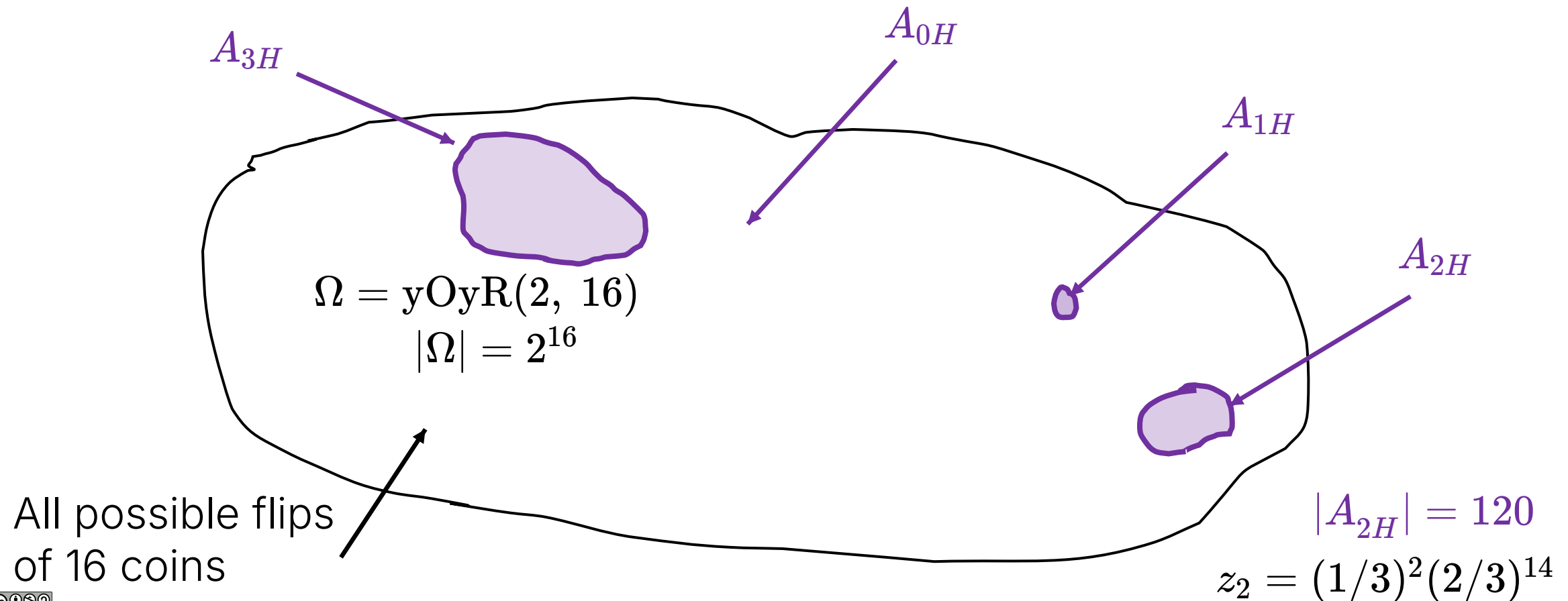
$$\mathbf{P}(A_{<4H}) = |A_{0H}| z_0 + \cdots + |A_{3H}| z_3, \quad \text{where } z_k = (1/3)^k (2/3)^{16-k}$$



$A_{<4H}$ as a Union of Events

Can also think of $A_{<4H}$ as $A_{0H} \cup A_{1H} \cup A_{2H} \cup A_{3H}$.

- Here $\mathbf{P}(A_{<4H}) = \mathbf{P}(A_{0H} \cup A_{1H} \cup A_{2H} \cup A_{3H}) = \mathbf{P}(A_{0H}) + \mathbf{P}(A_{1H}) + \mathbf{P}(A_{2H}) + \mathbf{P}(A_{3H})$
- We will revisit the probability of unions of events in the next lecture more carefully.



Poker Hands

Lecture 16, CS70 Fall 2026

Introduction

- Probability and CS70
- Probability Spaces
- Event Probability

Event Probability Examples

- Counting Sample Points
- Counting Weighted Sample Points
- **Poker Hands**
- Balls and Bins
- The Birthday Paradox
- Summary of Examples

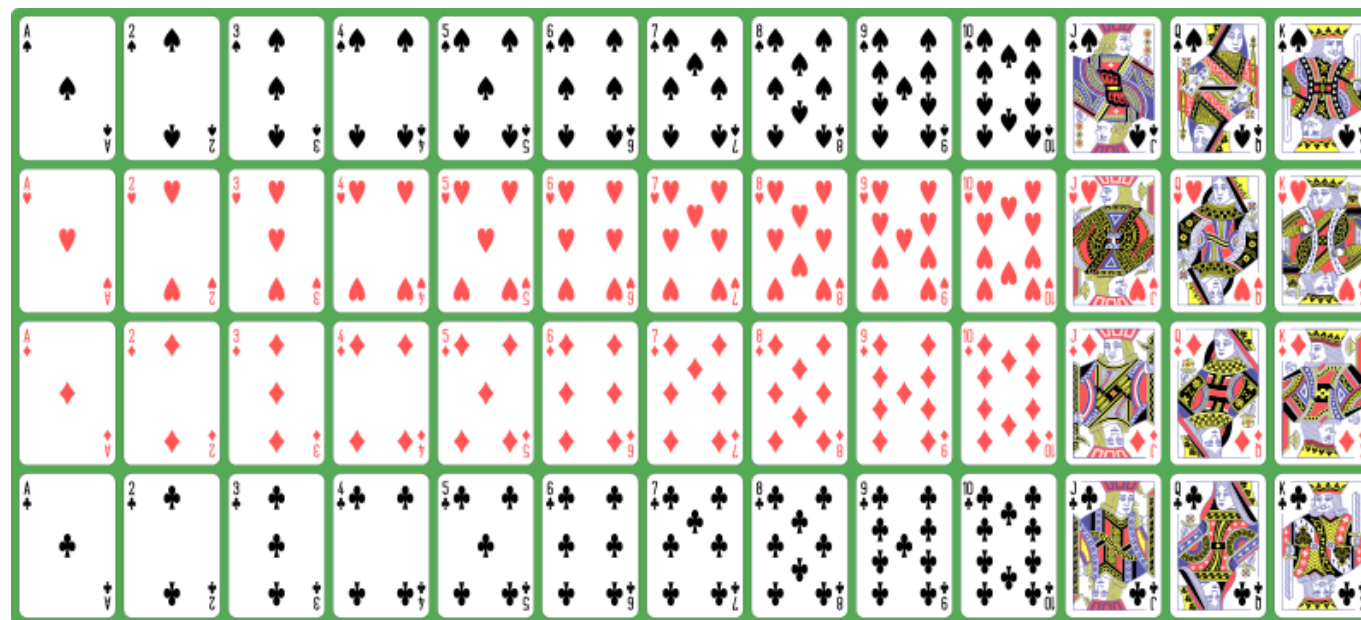
Playing Cards

In a standard deck of American playing cards, there are 52 cards.

- Each card has a "rank" and a "suit".
- Ranks: {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}
- Suits: {♠, ♥, ♦, ♣}

When shuffled, there are $52!$ (roughly 8×10^{67}) permutations.

- No two shuffled decks have ever been the same!



Poker Hands as a Random Experiment

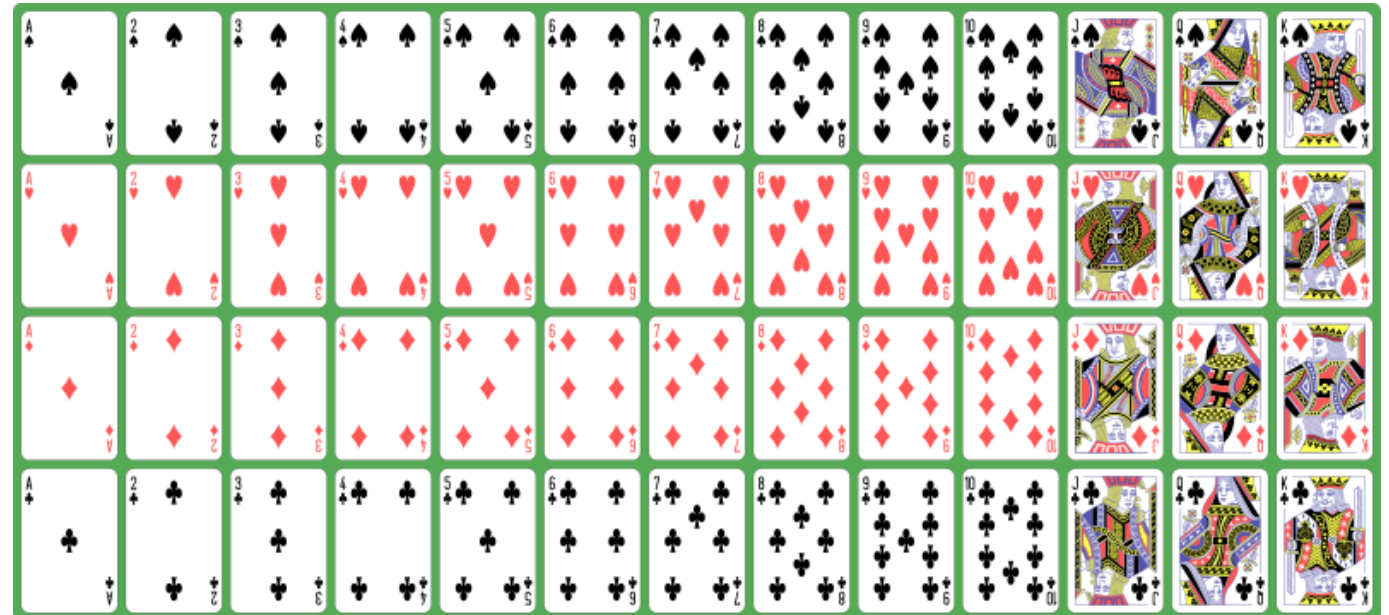
In the game of poker, each player is given a “hand” consisting of 5 cards from the set of 52 cards shown below.

- Each card has a “rank” and a “suit”.
- Ranks: {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}
- Suits: {♠, ♥, ♦, ♣}

Example Hand: {A♥, 2♣, 5♣, K♣, Q♦}

Questions:

- What is n (the cardinality of the set we’re drawing from)? What is k ? What is $|\Omega|$? Which counting problem?
- Is this with/without replacement? Order?
- Uniform probability space?



Poker Hands as a Random Experiment

In the game of poker, each player is given a “hand” consisting of 5 cards from the set of 52 cards shown below.

- Each card has a “rank” and a “suit”.
- Ranks: {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}
- Suits: {♠, ♥, ♦, ♣}

Example Hand: {A♥, 2♣, 5♣, K♣, Q♦}

Will see later we can use yOnR instead!



Questions:

- What is n (the cardinality of the set we’re drawing from)? What is k ? What is $|\Omega|$? Which counting problem?
- Is this with/without replacement? Order?
- Uniform probability space?

My answers: Ω is given by nOnR.

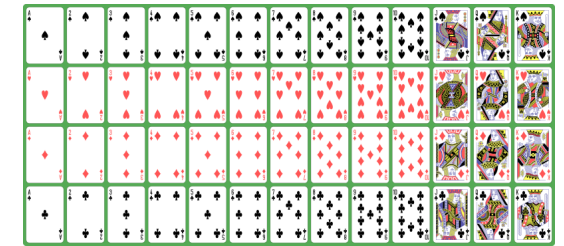
- n is 52. k is 5. Drawing without replacement. Ignoring order.
- $|\Omega|$: Drawing 5 items without replacement from a set of cardinality 52: $\binom{52}{5}$ or 2,598,960 possible hands.
- The space is uniform.

Poker Hands as a Random Experiment

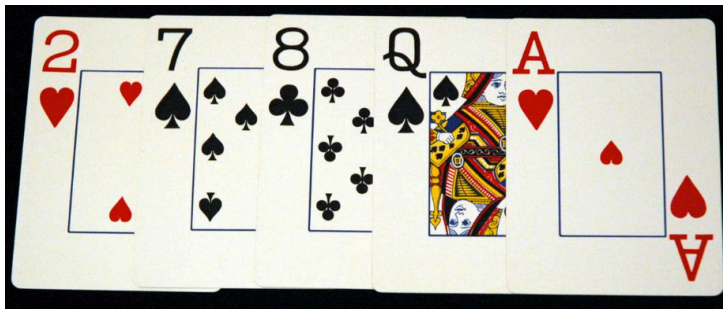
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- Each card has a "rank" and a "suit".
- Ranks: {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}
- Suits: {♠, ♥, ♦, ♣}

Example Hand: {A♥, 2♣, 5♣, K♣, Q♦}



Question: Which hand is more likely?

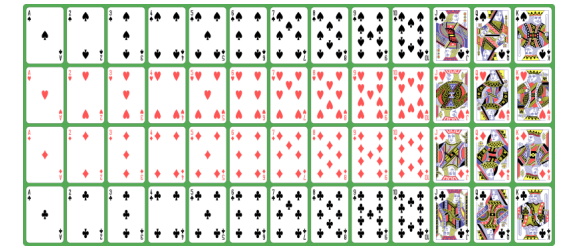


Poker Hands as a Random Experiment

In the game of poker, each player is given a "hand" consisting of 5 cards from the set of 52 cards shown below.

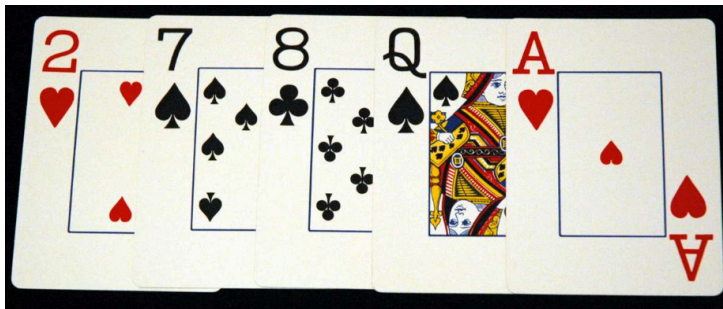
- Each card has a "rank" and a "suit".
- Ranks: {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}
- Suits: {♠, ♥, ♦, ♣}

Example Hand: {A♥, 2♣, 5♣, K♣, Q♦}



Question: Which hand is more likely?

- They are equally likely. It's a uniform probability space.



Poker Hands: Probability of a "Flush"

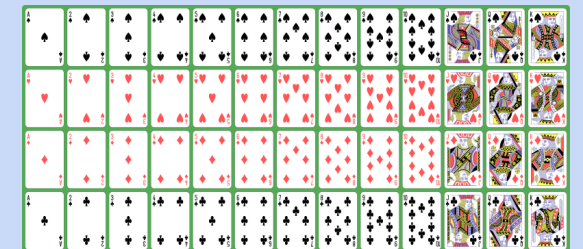
In the game of poker, each player is given a "hand" consisting of 5 cards from the set of 52 cards shown below.

Example Flushes: $\{A♥, 2♥, 5♥, K♥, Q♥\}$ $\{5♥, A♥, 2♥, Q♥, K♥\}$

Let A be the event that all five cards have the same suit (a "flush"). What is $P(A)$?

- This is a uniform probability space so we just need to compute $|A|/|\Omega|$.
- $|\Omega|$: Given by $nOnR(52, 5)$. Number of possible hands (ignoring order): $\binom{52}{5}$ or 2,598,960.
- $|A♥|$: Number of different ♥ flushes (ignoring order):

<https://joshh.ug/F>



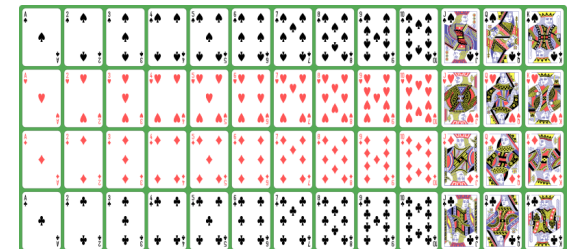
Poker Hands: Probability of a "Flush"

In the game of poker, each player is given a "hand" consisting of 5 cards from the set of 52 cards shown below.

Example Flushes: $\{A♥, 2♥, 5♥, K♥, Q♥\}$ $\{5♥, A♥, 2♥, Q♥, K♥\}$

Let A be the event that all five cards have the same suit (a "flush"). What is $P(A)$?

- This is a uniform probability space so we just need to compute $|A|/|\Omega|$.
- $|\Omega|$: Given by $nOnR(52, 5)$. Number of possible hands (ignoring order): $\binom{52}{5}$ or 2,598,960.
- $|A_♥|$: Number of different ♥ flushes ignoring order: Given by $nOnR(13, 5)$. Out of 13 hearts, how many ways to draw 5 hearts? Yields $\binom{13}{5} = 1287$ heart flushes.
- Four different suits, so total number of flushes = 5,148
- Thus, chance of a flush is $5,148/2,598,960 \approx 0.2\%$



In the game of poker, each player is given a “hand” consisting of 5 cards from the set of 52 cards shown below.

- Each card has a “rank” and a “suit”. Example Hand: {A♥, 2♣, 5♣, K♣, Q♦}
- Ranks: {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}
- Suits: {♠, ♥, ♦, ♣}

Questions:

- What is n (the cardinality of the set we’re drawing from)? What is k ? What is $|\Omega|$? Which counting problem?
- Is this with/without replacement? Order?
- Uniform probability space?

My answers: Ω is given by yOnR.

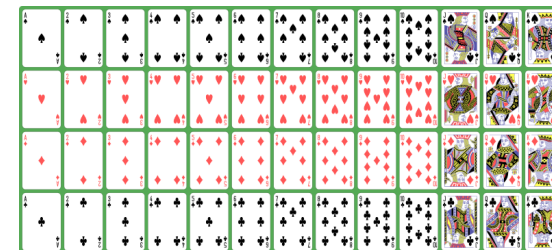
- n is 52. k is 5. Drawing without replacement. Considering order.
- $|\Omega|$: Drawing 5 items without replacement from a set of cardinality 52: $52 \times 51 \times 50 \times 49 \times 48$ or 311,875,200 possible hands.
- The space is uniform.

In the game of poker, each player is given a “hand” consisting of 5 cards from the set of 52 cards shown below.

Example Flushes: $\{A♥, 2♥, 5♥, K♥, Q♥\}$ $\{5♥, A♥, 2♥, Q♥, K♥\}$

Let A be the event that all five cards have the same suit (a “flush”). What is $P(A)$?

- This is a uniform probability space so we just need to compute $|A|/|\Omega|$.
- $|\Omega|$: Given by $yOnR(52, 5)$. Number of possible hands (taking into account order): $52 \times 51 \times 50 \times 49 \times 48$ or 311,875,200.
- $|A_♥|$: Number of different ♥ flushes:

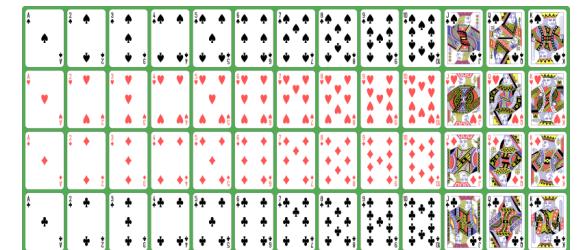


In the game of poker, each player is given a "hand" consisting of 5 cards from the set of 52 cards shown below.

Example Flushes: $\{A♥, 2♥, 5♥, K♥, Q♥\}$ $\{5♥, A♥, 2♥, Q♥, K♥\}$

Let A be the event that all five cards have the same suit (a "flush"). What is $P(A)$?

- This is a uniform probability space so we just need to compute $|A|/|\Omega|$.
- $|\Omega|$: Given by $yOnR(52, 5)$. Number of possible hands (taking into account order): $52 \times 51 \times 50 \times 49 \times 48$ or 311,875,200.
- $|A_{♥}|$: Number of different ♥ flushes: Given by $yOnR(13, 5)$. Out of 13 hearts, how many ways to draw 5 hearts? Yields $13 \times 12 \times 11 \times 10 \times 9$ or 154,440 heart flushes.
- Four different suits, so total number of flushes = 617,760
- Thus, chance of a flush is $617,760/311,875,200 \approx 0.2\%$



Observation: yOnR and nOnR Both Worked

Earlier, when drawing with replacement (e.g. rolling dice), it was important to consider order to ensure a uniform probability space.

By contrast, when drawing without replacement (e.g. drawing cards), it doesn't actually matter if you consider order.

Why?

- When drawing with replacement (e.g. dice rolls), ignoring order means samples with repeat draws are less likely if the individual draws are fair, e.g. $P(\{1, 1\}) < P(\{2, 1\})$.
- When drawing without replacement, this is not a concern. Can't get repeat draws in a single sample.

yOnR vs. nOnR Doesn't Matter When Drawing without Replacement

When drawing **without replacement**, it doesn't really matter if we choose to frame as yOnR or nOnR.

- **Both framings yield uniform probability spaces**, so calculations are equally easy.
- Recall that $|\text{nOnR}| = \frac{|\text{yOnR}|}{k!}$.
- Example: Computing the probability of a flush: $\mathbf{P}(A) = \frac{|A|}{|\Omega|}$
- Framing as yOnR basically adds an extra factor of $k!$ to both the numerator and denominator.

By contrast, when rolling dice (i.e. with replacement), we want to frame as yOyR and not nOyR to ensure a uniform probability space.

Balls and Bins

Lecture 16, CS70 Fall 2026

Introduction

- Probability and CS70
- Probability Spaces
- Event Probability

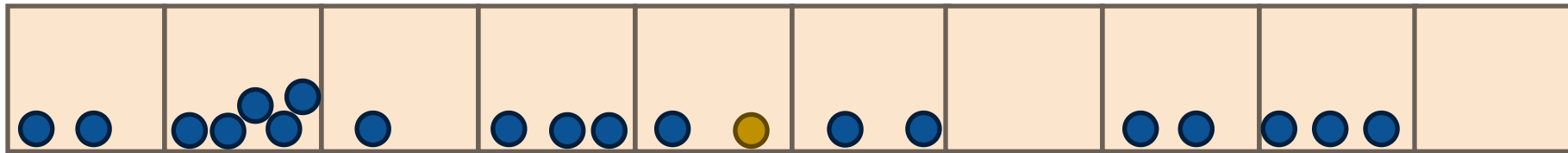
Event Probability Examples

- Counting Sample Points
- Counting Weighted Sample Points
- Poker Hands
- **Balls and Bins**
- The Birthday Paradox
- Summary of Examples

Example: Balls and Bins

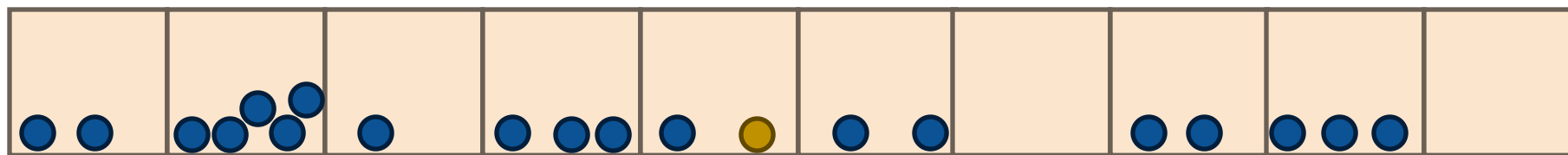
Suppose we throw 20 balls independently into 10 bins. Assume each ball is equally likely to land in any of the ten bins.

Let's do a quick demo.



Example: Balls and Bins

Suppose we throw 20 balls independently into 10 bins. Assume each ball is equally likely to land in any of the ten bins.



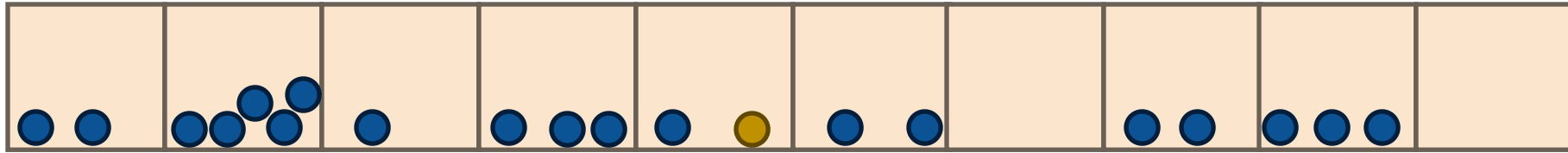
Suppose we want to be able to answer questions like: "What is $P(A_{bin1empty})$?"

- How should we define the sample space?
- How can we ensure our probability space is uniform?

(If watching the video, you might consider pausing here to think about this)

Example: Balls and Bins

Suppose we throw 20 balls independently into 10 bins. Assume each ball is equally likely to land in any of the ten bins.



My proposal: Treat just like 20 **ordered** dice throws (yOyR) of a ten-sided die.

- Example outcome: [5, 6, 2, 6, 8, 3, 9, 4, 9, 2, 1, 8, 2, 1, 9, 2, 2, 4, 4, 5]

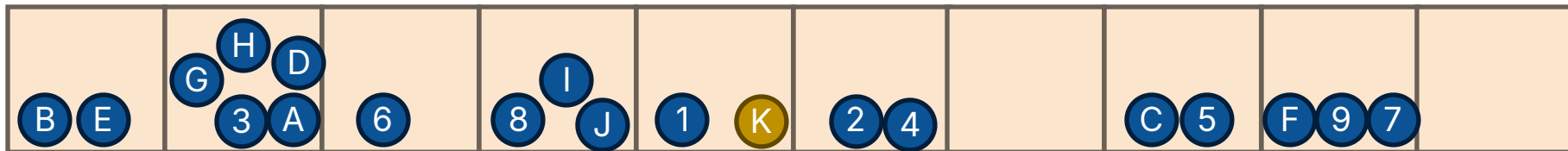
Example: Balls and Bins

Suppose we throw 20 balls independently into 10 bins. Assume each ball is equally likely to land in any of the ten bins.

- Framed as yOyR (ordered, w/replacement) with $k=20$, $n=10$, $|\Omega| = 10^{20}$

Note: By using yOyR as our framing, we've effectively labeled the balls.

- This is true even though the original problem had no labels.
- Ensures probability space is uniform.



Example: Balls and Bins

Suppose we throw 20 balls independently into 10 bins. Assume each ball is equally likely to land in any of the ten bins.

- $k=20$, $n=10$, with replacement, $|\Omega| = 10^{20}$

What is $\mathbf{P}(A_{bin1empty})$, i.e. the probability bin 1 has no ball?

- Since all outcomes are equally likely, just need to count number of outcomes where no ball lands in bin 1.
- Number of ways 20 balls can land in 9 bins:

Example: Balls and Bins

Suppose we throw 20 balls independently into 10 bins. Assume each ball is equally likely to land in any of the ten bins.

- $k=20$, $n=10$, with replacement, $|\Omega| = 10^{20}$

What is $P(A_{bin1empty})$, i.e. the probability bin 1 has no ball?

- Since all outcomes are equally likely, just need to count number of outcomes where no ball lands in bin 1.
- Number of ways 20 balls can land in 9 bins:

<https://joshh.ug/G>



Example: Balls and Bins

Suppose we throw 20 balls independently into 10 bins. Assume each ball is equally likely to land in any of the ten bins.

- $k=20$, $n=10$, with replacement, $|\Omega| = 10^{20}$

What is $\mathbf{P}(A_{bin1empty})$, i.e. the probability bin 1 has no ball?

- Since all outcomes are equally likely, just need to count number of outcomes where no ball lands in bin 1.
- Number of ways 20 balls can land in 9 bins: 9^{20}
 - Can think of this as "Assume that the first bin is empty, how many different configurations are there?"
- So $\mathbf{P}(A_{bin1empty}) = 9^{20} / 10^{20} \approx 12\%$

Example: Balls and Bins

Suppose we throw 20 balls independently into 10 bins. Assume each ball is equally likely to land in any of the ten bins.

- $k=20$, $n=10$, with replacement, $|\Omega| = 10^{20}$

What is $\mathbf{P}(A_{bin1empty})$, i.e. the probability bin 1 has no ball?

- $9^{20}/10^{20} \approx 12\%$

What is $\mathbf{P}(A_{bin1nonempty})$, i.e. the probability at least one ball lands in bin 1?

- This is just $1 - \mathbf{P}(A_{bin1empty}) = 1 - 9^{20}/10^{20} \approx 88\%$

In general, for an event A , $P(A) = 1 - P(\overline{A})$

Example: Balls and Bins

Suppose we throw k balls independently into n bins. Assume each ball is equally likely to land in any of the n bins.

- k balls, n bins, with replacement, $|\Omega| = n^k$

What is $\mathbf{P}(A_{bin1empty})$, i.e. the probability bin 1 has no ball?

- For $k = 20$, $n = 10$, we had $\mathbf{P}(A_{bin1empty}) = 9^{20}/10^{20}$
- For general k and n , we have $\mathbf{P}(A_{bin1empty}) = (n-1)^k/n^k$

$$\text{So } \mathbf{P}(A_{bin1empty}) = \frac{(n-1)^k}{n^k} = \left(\frac{n-1}{n}\right)^k = \left(1 - \frac{1}{n}\right)^k$$

Number of ways k balls
can land in $n-1$ bins:
 $(n-1)^k$

Example: Balls and Bins (Skipped: Extra Slide)

$$\text{So } \mathbf{P}(A_{bin1empty}) = \frac{(n-1)^k}{n^k} = \left(\frac{n-1}{n}\right)^k = \left(1 - \frac{1}{n}\right)^k$$

Another way to arrive at this equation is to consider that the chance of landing in some bin other than bin 1 is $(1 - \frac{1}{n})$.

Chance of throwing two balls and neither landing in bin 1 is $(1 - \frac{1}{n})^2$.

Chance of throwing k balls and none landing in bin 1 is $(1 - \frac{1}{n})^k$.

Buckets and Balls and the World

Suppose we throw k balls independently into n bins. Assume each ball is equally likely to land in any of the n bins.

This problem is a common model in computer science, e.g. modeling jobs being sent to one of many servers. Suppose k jobs are sent to one of n servers randomly.

- $\mathbf{P}(A_{server1notused}) = \left(\frac{n-1}{n}\right)^k = \left(1 - \frac{1}{n}\right)^k$

Or k items are stored in one of n hash table buckets:

- $\mathbf{P}(A_{bucket1notused}) = \left(\frac{n-1}{n}\right)^k = \left(1 - \frac{1}{n}\right)^k$

The Birthday Paradox

Lecture 16, CS70 Fall 2026

Introduction

- Probability and CS70
- Probability Spaces
- Event Probability

Event Probability Examples

- Counting Sample Points
- Counting Weighted Sample Points
- Poker Hands
- Balls and Bins
- **The Birthday Paradox**
- Summary of Examples

Suppose we have 50 people in a room. What is the chance that two share a birthday? (For simplicity, ignore leap year birthdays, i.e. assume each year has 365 days).

What's your gut feeling?

- Less than 20%?
- Decent chance, say 50%ish?
- Very likely, say >90%?

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365.

What is the probability space for the possible birthdays?

- What is the cardinality n of the set S that we're drawing from?
- How many samples k are we drawing?
- Are we drawing with or without replacement?
- Are we considering order?
- What is the cardinality of the sample space Ω ?

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365.

What is the probability space for the possible birthdays? This is yOyR.

- What is the cardinality n of the set S that we're drawing from? 365
- How many samples k are we drawing? 50
- Are we drawing with or without replacement? With
- Are we considering order? Yes (uniform probability space)
- What is the cardinality of the sample space Ω ? $|\Omega| = 365^{50}$

Note: This is just yOyR again. Same as balls-and-bins with 50 balls and 365 bins. Or rolling a 365-sided die 50 times.

Birthdays

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365.

- Cardinality of the sample space Ω ? $|\Omega| = 365^{50}$

Let A be the event where two (or more) of the birthdays are the same. Then:

$$P(A) = \frac{|A|}{|\Omega|}$$

Examples from A : $[1, 1, \dots]$, $[5, 5, \dots 5, \dots]$, $[66, 12, 51, 66, \dots]$

Problem: How the heck do we figure out $|A|$?

Birthdays

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365.

- Cardinality of the sample space Ω ? $|\Omega| = 365^{50}$

Let A be the event where two (or more) of the birthdays are the same. Hard to compute $|A|$.

As is sometimes the case, $|\overline{A}|$ is easier to compute.

- Example from $|\overline{A}|$: [55, 1, 67, 2, 99, ..., 33, 41], with no repeats.

How many samples are there like this with no repeats?

- Let's think about this as a different counting problem.

Birthdays with no Repeats

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365 **AND THERE ARE NO REPEAT BIRTHDAYS.**

Goal: Figure out the size of this sample space.

- What is the cardinality n of the set S that we're drawing from?
- How many samples k are we drawing?
- Are we drawing with or without replacement?
- What is the cardinality of this sample space?

Birthdays with no Repeats

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365 **AND THERE ARE NO REPEAT BIRTHDAYS.**

Goal: Figure out the size of this sample space.

- What is the cardinality n of the set S that we're drawing from? **365**
- How many samples k are we drawing? **50**
- Are we drawing with or without replacement? **Without**
- What is the cardinality of this sample space?

<https://joshh.ug/H>



Birthdays with no Repeats

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365 **AND THERE ARE NO REPEAT BIRTHDAYS.**

Goal: Figure out the size of this sample space.

- What is the cardinality of the set S that we're drawing from? **365**
- How many samples are we drawing? **50**
- Are we drawing with or without replacement? **Without**
- What is the cardinality of this sample space? $365 \times 364 \times \cdots \times 316$

Note that this sample space is just $\overline{A}$, i.e. $|\overline{A}| = 365 \times 364 \times \cdots \times 316$

Birthdays

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365. What is the chance that two of those 50 people share the same birthday?

Let A be the event where two (or more) of the birthdays are the same. Then:

$$P(A) = \frac{|A|}{|\Omega|} = 1 - P(\overline{A}) = 1 - \frac{|\overline{A}|}{|\Omega|}$$

For $k = 50$, $|\overline{A}| = 365 \times 364 \times \cdots \times 316$, and $|\Omega| = 365^{50}$, so

$$P(A) = 1 - \frac{365 \times 364 \times \cdots \times 316}{365^{50}}$$

Can use e.g.
[Wolfram|Alpha](#)
to compute.

Birthdays

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365. What is the chance that two of those 50 people share the same birthday?

Let A be the event where two (or more) of the birthdays are the same. Then:

$$P(\textcolor{red}{A}) = \frac{|\textcolor{red}{A}|}{|\Omega|} = 1 - P(\overline{\textcolor{blue}{A}}) = 1 - \frac{|\overline{\textcolor{blue}{A}}|}{|\Omega|}$$

For $k = 50$, $|\overline{\textcolor{blue}{A}}| = 365 \times 364 \times \cdots \times 316$, and $|\Omega| = 365^{50}$, so

$$P(\textcolor{red}{A}) = 1 - \frac{365 \times 364 \times \cdots \times 316}{365^{50}} \approx 1 - 0.03 \approx \textcolor{red}{0.97}$$

Birthdays

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365. What is the chance that two of those 50 people share the same birthday?

- Compute by counting number of birthday combinations with no repeats.

$$|\overline{A}| = 365 \times 364 \times \cdots \times 316$$

Suppose we want to generalize to the case where we have k people. What is $|\overline{A}|$ in that case?

$$P(\textcolor{red}{A}) = 1 - \frac{365 \times 364 \times \cdots \times 316}{365^{50}} \approx 1 - 0.03 \approx \textcolor{red}{0.97}$$

Birthdays

Suppose we have 50 people in a room who all have a birthday between day 1 and day 365. What is the chance that two of those 50 people share the same birthday?

- Compute by counting number of birthday combinations with no repeats.

$$|\overline{A}| = 365 \times 364 \times \cdots \times 316$$

Suppose we want to generalize to the case where we have k people. What is $|\overline{A}|$ in that case?

$$|\overline{A}| = 365 \times 364 \times \cdots \times (365 - k + 1)$$

Birthdays

Suppose we have k people in a room who all have a birthday between day 1 and day 365. What is the chance that two of those k people share the same birthday?

$$P(A) = 1 - \frac{365 \times 364 \times \cdots \times (365 - k + 1)}{365^k}$$

Note: Sources use different letters for these. We write k for the number of people and $n = 365$ for the number of days, matching $yOyR(n, k)$. The course notes (Note 15) use n for the number of people and just write 365 for the days; Wikipedia uses n people and d days.

Quick Aside: Computing This Function

Suppose we want to compute the results of the function below.

- Why might it be a bad idea to compute the numerator and denominator separately?
- How can we compute the result more efficiently?

$$P(A) = 1 - \frac{365 \times 364 \times \cdots \times (365 - k + 1)}{365^k}$$

Quick Aside: Computing This Function

Suppose we want to compute the results of the function below.

- Why might it be a bad idea to compute the numerator and denominator separately?
 - In many programming languages like Java, we'd get overflow issues.
 - In languages that support huge integers, performance will be slow. 365^{50} is a 426 bit number, much bigger than 64 bits for typical operations.
- How can we compute the result more efficiently?
 - One idea: Compute product of $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \dots$
 - You can compute this approximately in constant time (maybe try to come up with a technique)

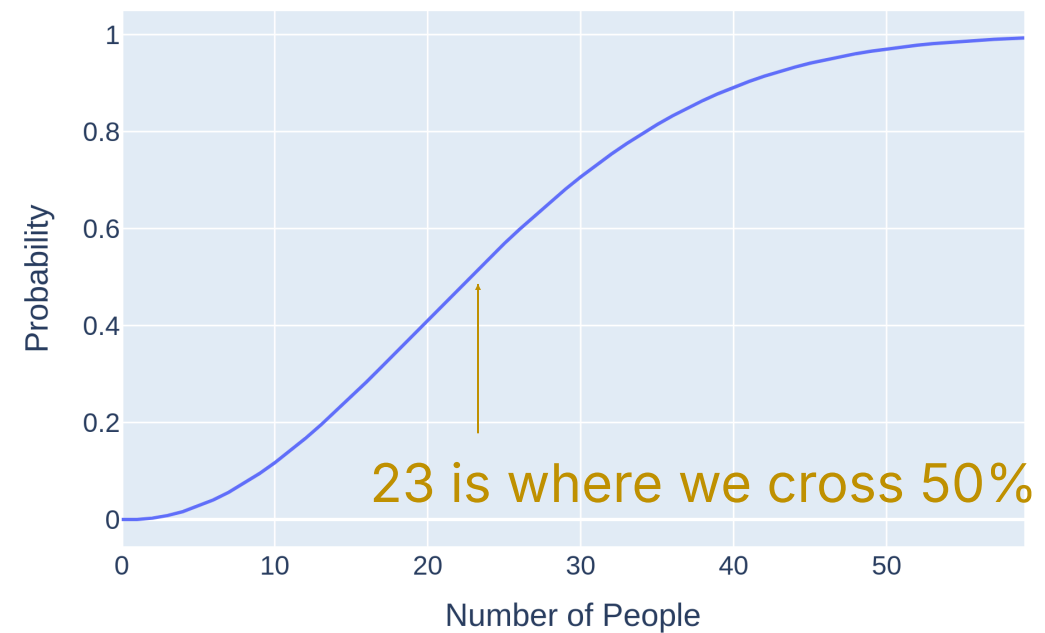
$$P(A) = 1 - \frac{365 \times 364 \times \dots \times (365 - k + 1)}{365^k}$$

Birthday Paradox Table

For various k , the probability that at least two people share a birthday is given below.

k	$P(A)$	%
1	0	0%
2	0.0027	0.27%
3	0.008	0.8%
4	0.016	1.6%
10	0.117	11.7%
15	0.252	25.2%
20	0.411	41.1%
23	0.507	50.7%
50	0.97	97.0%
60	0.994	99.4%

Probability of Birthday Match as k Increases



Summary of Examples

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- The Birthday Paradox
- **Summary of Examples**

Coins, Dice, Balls & Bins, and Birthdays

Flipping coins, rolling dice, tossing balls into bins, and picking birthdays are all the same problem: $\text{yOyR}(n, k)$, where n is the cardinality of the set we're drawing from, and k is the number of samples drawn.

- Rolling a 4 sided die twice: $k = 2, n = 4$.
- Flipping a coin 16 times: $k = ?, n = ?$.

Coins, Dice, Balls & Bins, and Birthdays

Flipping coins, rolling dice, tossing balls into bins, and picking birthdays are all the same problem: $\text{yOyR}(n, k)$, where n is the cardinality of the set we're drawing from, and k is the number of samples drawn.

- Rolling a 4 sided die twice: $k = 2, n = 4$.
- Flipping a coin 16 times: $k = 16, n = 2$.
- Tossing 20 balls in 10 bins: $k = ?, n = ?$.

Coins, Dice, Balls & Bins, and Birthdays

Flipping coins, rolling dice, tossing balls into bins, and picking birthdays are all the same problem: $\text{yOyR}(n, k)$, where n is the cardinality of the set we're drawing from, and k is the number of samples drawn.

- Rolling a 4 sided die twice: $k = 2, n = 4$.
- Flipping a coin 16 times: $k = 16, n = 2$.
- Tossing 20 balls in 10 bins: $k = 20, n = 10$.
- Picking 23 birthdays: $k = ?, n = ?$.

Coins, Dice, Balls & Bins, and Birthdays

Flipping coins, rolling dice, tossing balls into bins, and picking birthdays are all the same problem: $\text{yOyR}(n, k)$, where n is the cardinality of the set we're drawing from, and k is the number of samples drawn.

- Rolling a 4 sided die twice: $k = 2, n = 4$.
- Flipping a coin 16 times: $k = 16, n = 2$.
- Tossing 20 balls in 10 bins: $k = 20, n = 10$.
- Picking 23 birthdays: $k = 23, n = 365$.

Example: Chance of getting the same die roll in 23 rolls of a 365-sided die is around 50%.

Coins, Dice, Balls & Bins, and Birthdays

Flipping coins, rolling dice, tossing balls into bins, and picking birthdays are all the same problem: $yOyR(n, k)$, where n is the cardinality of the set we're drawing from, and k is the number of samples drawn.

- Rolling a 4 sided die twice: $k = 2, n = 4$.
- Flipping a coin 16 times: $k = 16, n = 2$.
- Tossing 20 balls in 10 bins: $k = 20, n = 10$.
- Picking 23 birthdays: $k = 23, n = 365$.

Drawing cards, picking the position for heads, or picking birthdays without repeats are all the same problem: $nOnR(n, k)$ [though you can also use $yOnR$]:

- Drawing 5 cards from a deck of 52: $k = 5, n = 52$.
- Figuring out how many ways to get 6 heads out of 16 flips: $k = 6, n = 16$.
- Picking 23 birthdays without repeats: $k = 23, n = 365$.

Summary of Lecture 15 Topic Appearances Today

Three of our four counting problems from lecture 15 were useful today:

- $yOnR(n, k)$: poker hands with order, $|\Omega| = 52 \times 51 \times 50 \times 49 \times 48$; birthdays with no repeats, $|\overline{A}| = 365 \times 364 \times \cdots \times (365 - k + 1)$.
- $yOyR(n, k)$: the sample space Ω for coins, dice, balls in bins, and birthdays.
- $nOnR(n, k)$: poker hands, $|\Omega| = \binom{52}{5}$; number of ways to get 6 heads $|A_{6H}| = \binom{16}{6}$.
- $nOyR(n, k)$: never used as Ω , because it isn't uniform: for two fair 4-sided dice, $P(\{1, 1\}) = 1/16$ but $P(\{1, 2\}) = 1/8$.

Summary of Lecture 16

Probability is hard!

- Intuitions can be wildly wrong.
- Rigor is critical for correctness.

Recall the key steps in all our calculations (framed slightly differently):

- Determine Ω , the set of all possible outcomes. Favor $yOyR$ over $nOyR$.
- Determine $P(\omega)$, the probability of each outcome $\omega \in \Omega$.
 - For uniform probability spaces, $P(\omega) = 1/|\Omega|$
- Determine the event A that we're interested in.
- Compute the probability of A by adding up the probabilities of the sample points contained in it. For uniform spaces, $P(A) = |A|/|\Omega|$
 - Often involves counting the size of A .
 - Sometimes the complement of an event is easier to count.