

The Principle of Inclusion-Exclusion

Lecture 18, CS70 Fall 2026

The Principle of Inclusion-Exclusion

Combinations of Events

- Independence
- Intersections and The Product Rule
- Product Rule Applications and Proof
- Unions of Events
- Unions of Events (Large N)
- Summary

The Size of the Union of Two Sets

Suppose we have two sets A_1 and A_2 .

We know that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

- $|A_1 \cup A_2|$: The size of their union.
- $|A_1| + |A_2|$: The sum of their individual sizes.
- $|A_1 \cap A_2|$: The size of their intersection.

Example: $A_1 = \{a, b, c, d\}$, $A_2 = \{c, d, e, f\}$

The Size of the Union of Two Sets

Suppose we have two sets A_1 and A_2 .

We know that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

- $|A_1 \cup A_2|$: The size of their union.
- $|A_1| + |A_2|$: The sum of their individual sizes.
- $|A_1 \cap A_2|$: The size of their intersection.

Example: $A_1 = \{a, b, c, d\}$, $A_2 = \{c, d, e, f\}$ $|A_1 \cup A_2| = 4 + 4 - 2$

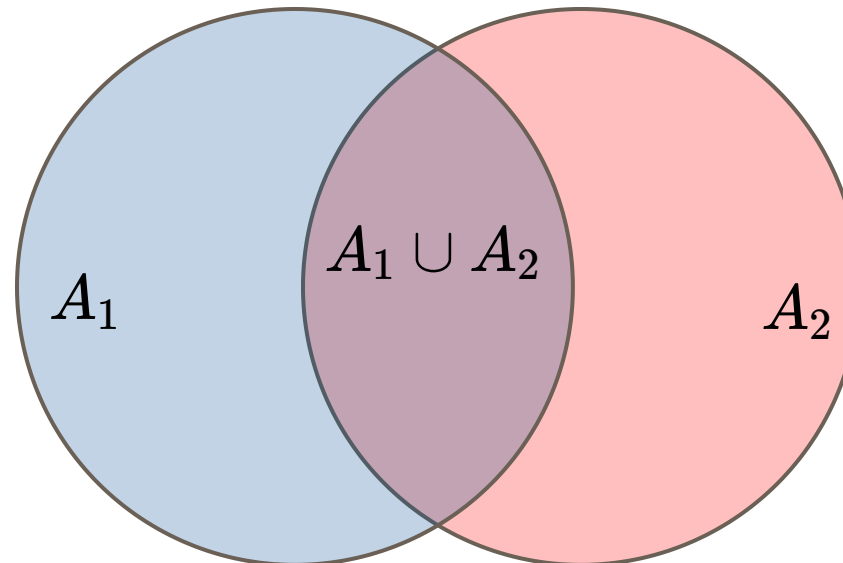
- $|A_1 \cup A_2| = |\{a, b, c, d, e, f\}| = 6$
- $|A_1| + |A_2| = 4 + 4$
- $|A_1 \cap A_2| = |\{c, d\}| = 2$

Set Union Size (Visually)

Suppose we have two sets A_1 and A_2 .

We know that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

- $|A_1 \cup A_2|$: The size of their union.
- $|A_1| + |A_2|$: The sum of their individual sizes.
- $|A_1 \cap A_2|$: The size of their intersection.



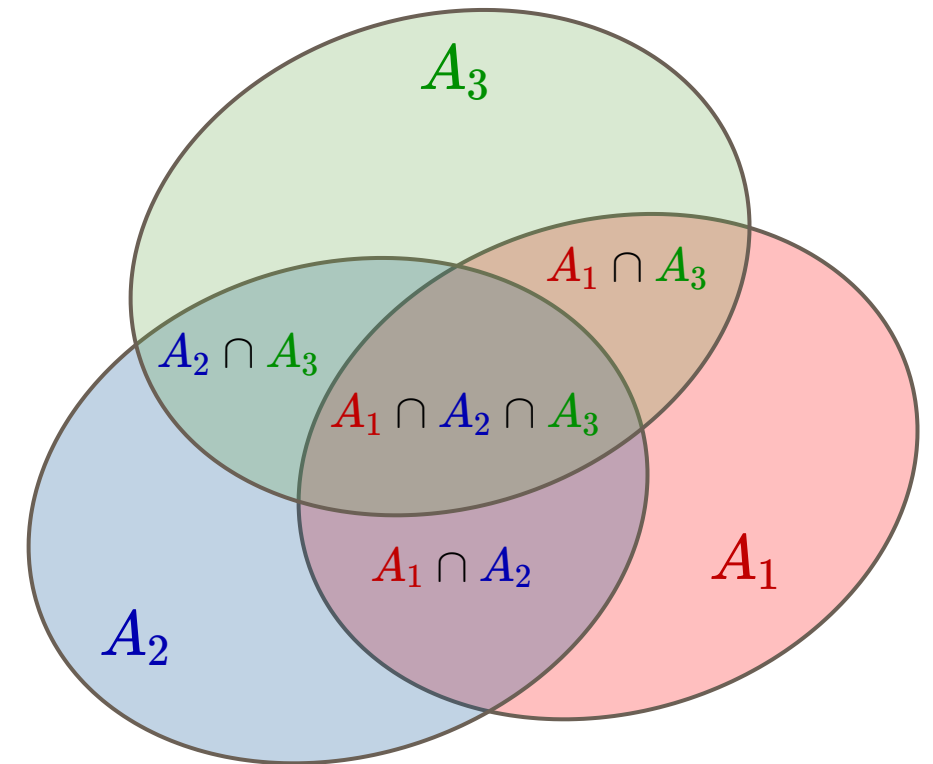
For Three Sets

Suppose we have three sets A_1 , A_2 , and A_3 .

We might first believe that $|A_1 \cup A_2 \cup A_3|$ is given by:

$$|A_1 \cup A_2 \cup A_3| \stackrel{??}{=} |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$$

What's wrong with this and how do we fix it?



For Three Sets

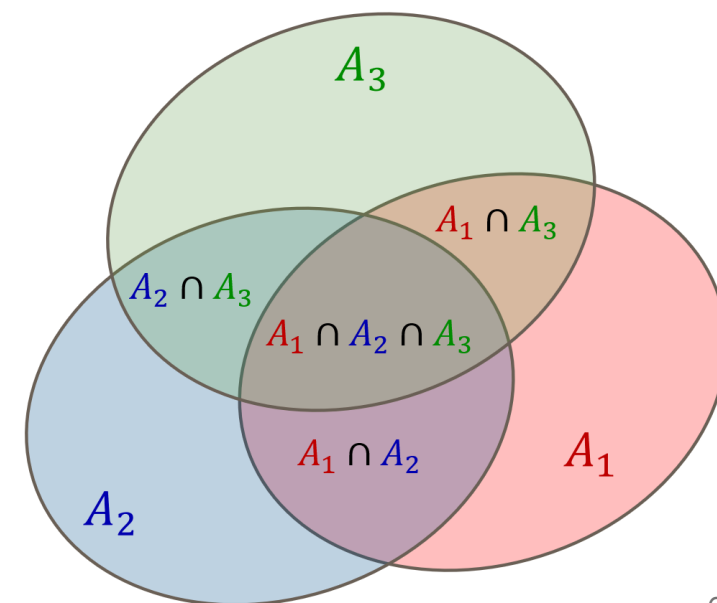
Suppose we have three sets A_1 , A_2 , and A_3 .

We might first believe that $|A_1 \cup A_2 \cup A_3|$ is given by:

$$|A_1 \cup A_2 \cup A_3| \stackrel{??}{=} |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$$

What's wrong with this and how do we fix it?

- $|A_1| + |A_2| + |A_3|$ triple counts $A_1 \cap A_2 \cap A_3$ (at center).
- Then we subtract out these points three times.
- Uh-oh! Items in $A_1 \cap A_2 \cap A_3$ get counted zero times.
- To fix, we just add back $|A_1 \cap A_2 \cap A_3|$.

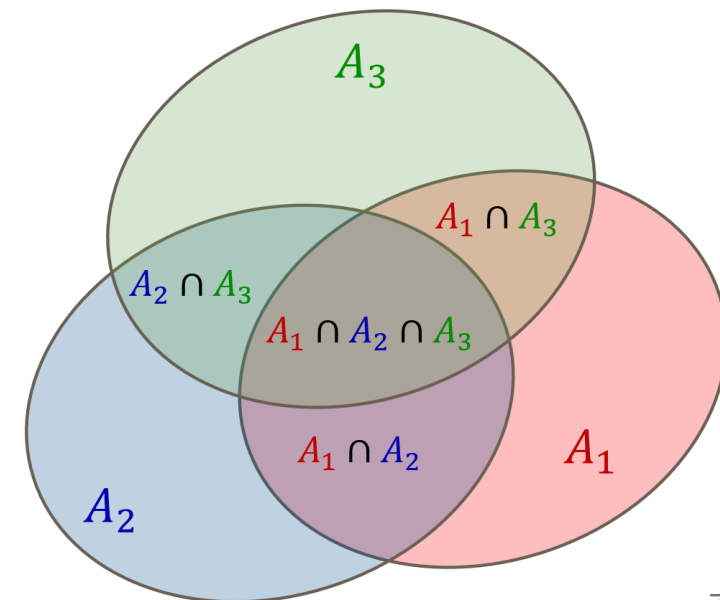


For Three Sets

Suppose we have three sets A_1 , A_2 , and A_3 . Then the size of their union is:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

This is known as the “Principle of Inclusion-Exclusion”.



Alternate Justification for the PIE Formula

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Another way to understand this formula:

- Consider a point $x \in A_1 \cap A_2 \cap A_3$, i.e. x belongs to all three sets.
 - x counted once in each of: $|A_1|$, $|A_2|$, and $|A_3|$, or 3 times total.
 - x counted once in each of: $|A_1 \cap A_2|$, $|A_2 \cap A_3|$, and $|A_1 \cap A_3|$, or 3 times total.
 - x counted once in $|A_1 \cap A_2 \cap A_3|$.
- Can also think of $x \in A_1 \cup A_2 \cup A_3$ as:
 - Belonging to $\binom{3}{1} = 3$ non-intersections.
 - Belonging to $\binom{3}{2} = 3$ pairwise intersections.
 - Belonging to $\binom{3}{3} = 1$ three-way intersection.
 - Counted exactly once by PIE formula: $3 - 3 + 1 = 1$

What if We Have Four Sets?

We can keep doing this. Without proof (yet), I offer the following procedure:

- With four sets, we add all the sizes together.
- But that results in an overcount, so we subtract out the sizes of the pairwise intersections.
- But that results in an undercount, so we add back in the sizes of the three-way intersections.
- But that results in an overcount, so we subtract the size of the four-way intersection.

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &\quad + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\ &\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

A More Compact Formula

Suppose we have 4 sets A_1 through A_4 . Then

$$|A_1 \cup \dots \cup A_4| = \sum_{k=1}^4 (-1)^{k-1} \text{INTERSECTSUM}_k(A_1, \dots, A_4)$$

Where $\text{INTERSECTSUM}_k(A_1, \dots, A_4)$ is the sum of the sizes of all k -way intersections of the sets. For example:

$$\text{INTERSECTSUM}_2(A_1, \dots, A_4) =$$

$$\text{INTERSECTSUM}_3(A_1, \dots, A_4) =$$

A More Compact Formula

Suppose we have 4 sets A_1 through A_4 . Then

$$|A_1 \cup \dots \cup A_4| = \sum_{k=1}^4 (-1)^{k-1} \text{INTERSECTSUM}_k(A_1, \dots, A_4)$$

Where $\text{INTERSECTSUM}_k(A_1, \dots, A_4)$ is the sum of the sizes of all k -way intersections of the sets. For example:

$$\begin{aligned} \text{INTERSECTSUM}_2(A_1, \dots, A_4) &= |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| \\ &\quad + |A_2 \cap A_4| + |A_3 \cap A_4| \end{aligned}$$

$$\begin{aligned} \text{INTERSECTSUM}_3(A_1, \dots, A_4) &= |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\ &\quad + |A_2 \cap A_3 \cap A_4| \end{aligned}$$

A More Compact Formula

Suppose we have 4 sets A_1 through A_4 . Then

$$\begin{aligned} |A_1 \cup \dots \cup A_4| &= \sum_{k=1}^4 (-1)^{k-1} \text{INTERSECTSUM}_k(A_1, \dots, A_4) \\ &= |A_1| + |A_2| + |A_3| + |A_4| \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &\quad + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\ &\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

We have not proven this yet! First, let's generalize.

General PIE For Any Number of Sets

Suppose we have n sets A_1 through A_n . Then

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(A_1, \dots, A_n)$$

Now let's prove this.

- Goal: Show that every element gets counted exactly once.
- We'll develop together a very clever combinatorial proof.
 - Sub-goal: Prove that PIE correctly counts items that belong to exactly 4 sets.
 - Then: Prove later that PIE counts items that belong to any number of sets.

PIE Combinatorial Proof (Warmup Question)

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

How many terms does $\text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$ have?

PIE Combinatorial Proof (Warmup Question)

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

How many terms does $\text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$ have? $\binom{n}{k}$

Example: If $n = 4$ and $k = 2$, we have:

PIE Combinatorial Proof (Warmup Question)

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

How many terms does $\text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$ have? $\binom{n}{k}$

Example: If $n = 4$ and $k = 2$, we have:

$$|\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2| + |\textcolor{red}{A}_1 \cap \textcolor{green}{A}_3| + |\textcolor{red}{A}_1 \cap A_4| + |\textcolor{blue}{A}_2 \cap \textcolor{green}{A}_3| + |\textcolor{blue}{A}_2 \cap A_4| + |\textcolor{green}{A}_3 \cap A_4|$$

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 . How many times is it counted in $\text{INTERSECTSUM}_1, \text{INTERSECTSUM}_2, \dots, \text{INTERSECTSUM}_{54}$?

How many times is x counted in each INTERSECTSUM_k ?

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in $\text{INTERSECTSUM}_1(\textcolor{red}{A}_1, \dots, A_n)$?

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in $\text{INTERSECTSUM}_1(\textcolor{red}{A}_1, \dots, A_n)$?

$$\text{INTERSECTSUM}_1(\textcolor{red}{A}_1, \dots, A_n) = |\textcolor{red}{A}_1| + |\textcolor{blue}{A}_2| + |\textcolor{green}{A}_3| + \dots + |A_n|$$

x is counted four times, because it belongs to exactly 4 of the sets above.

- This means x is being quadruple counted.

Note: INTERSECTSUM_1 is not the number of times x is counted, i.e. INTERSECTSUM_1 is not 4. It's probably some other bigger number.

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in INTERSECTSUM_2 ?

Note: To avoid wasting space, the following slides say INTERSECTSUM_k instead of $\text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in INTERSECTSUM_2 ?

$$\text{INTERSECTSUM}_2 = |\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2| + |\textcolor{red}{A}_1 \cap \textcolor{green}{A}_3| + |\textcolor{red}{A}_1 \cap A_4| + \dots + |A_{n-1} \cap A_n|$$

x appears in $\binom{4}{2} = 6$ terms. So x is counted 6 times in INTERSECTSUM_2 .

- That means INTERSECTSUM_1 quadruple-counted x , but now I've subtracted 6, so now I'm undercounting by 2.

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in INTERSECTSUM_3 ?

$$\text{INTERSECTSUM}_3 = |\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2 \cap \textcolor{green}{A}_3| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n|$$

In other words, in how many 3-way intersection sets $\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2 \cap \textcolor{green}{A}_3$, $\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2 \cap \textcolor{green}{A}_4$, $\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2 \cap \textcolor{green}{A}_5$, ... does x appear?

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in INTERSECTSUM_3 ?

$$\text{INTERSECTSUM}_3 = |\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2 \cap \textcolor{green}{A}_3| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n|$$

x appears in $\binom{4}{3} = 4$ terms. So x is counted 4 times in INTERSECTSUM_3 .

- First we quadruple counted x , then subtracted 6, now we added 4. So we're still double counting x since $4 - 6 + 4 = 2$.

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in INTERSECTSUM_4 ?

$$\text{INTERSECTSUM}_4 = |\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2 \cap \textcolor{green}{A}_3 \cap A_4| + \dots + |A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n|$$

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\mathbf{A_1} \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\mathbf{A_1}, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\mathbf{A_1}$, $\mathbf{A_3}$, A_6 , and A_8 .

How many times is x counted in INTERSECTSUM_4 ?

$$\text{INTERSECTSUM}_4 = |\mathbf{A_1} \cap \mathbf{A_2} \cap \mathbf{A_3} \cap A_4| + \dots + |A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n|$$

x appears in $\binom{4}{4} = 1$ term, i.e. the only four-way intersection to which it belongs is $|\mathbf{A_1} \cap \mathbf{A_3} \cap A_6 \cap A_8|$. So x is counted 1 time in INTERSECTSUM_4 .

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in INTERSECTSUM_5 ?

$$\text{INTERSECTSUM}_5 = |\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2 \cap \textcolor{green}{A}_3 \cap A_4 \cap A_5| + \dots$$

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

How many times is x counted in INTERSECTSUM_5 ?

$$\text{INTERSECTSUM}_5 = |\textcolor{red}{A}_1 \cap \textcolor{blue}{A}_2 \cap \textcolor{green}{A}_3 \cap A_4 \cap A_5| + \dots$$

x appears in $\binom{4}{5} = 0$ terms. That is x appears in no 5-way intersections.

Proof that PIE Correctly Counts Items Belonging to 4 Sets

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose we have 54 sets. Suppose x belongs to only sets $\textcolor{red}{A}_1$, $\textcolor{green}{A}_3$, A_6 , and A_8 .

k	# times counted in INTERSECTSUM_k
1	4
2	6
3	4
4	1
5	0
...	0

$$\begin{array}{rcl} 4 \times (-1)^0 & & 4 \\ + 6 \times (-1)^1 & & -6 \\ + 4 \times (-1)^2 & & +4 \\ + 1 \times (-1)^3 & & -1 \end{array}$$

Thus, we see that for any x that belongs to exactly 4 sets out of 54, our formula counts that item exactly once.

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

We showed that the formula correctly counts any x that belongs to 4 sets. To generalize, we observe the pattern in the # of times x is counted for each k .

k	# times counted in INTERSECTSUM_k	# times counted in INTERSECTSUM_k	
1	4	$\binom{4}{1}$	$(-1)^0$
2	6	$\binom{4}{2}$	$(-1)^1$
3	4	$\binom{4}{3}$	$(-1)^2$
4	1	$\binom{4}{4}$	$(-1)^3$

Table for an item x that belongs to 4 sets

PIE Combinatorial Proof

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose x belongs to m sets. Then the number of times x will be counted is:

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose x belongs to m sets. Then the number of times x will be counted is:

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \dots + \binom{m}{m} (-1)^{m-1}$$

We saw before that if we pick $m = 4$, we get $4 - 6 + 4 - 1$.

If you pick $m = 5$, we get $\binom{5}{1} - \binom{5}{2} + \binom{5}{3} - \binom{5}{4} + \binom{5}{5} = 5 - 10 + 10 - 5 + 1 = 1$

This means that if an item belongs to exactly 5 sets, it also gets counted correctly.

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose x belongs to m sets. Then the number of times x will be counted is:

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \dots + \binom{m}{m} (-1)^{m-1}$$

We saw before that if we pick $m = 4$, we get $4 - 6 + 4 - 1$.

If you pick $m = 5$, we get $\binom{5}{1} - \binom{5}{2} + \binom{5}{3} - \binom{5}{4} + \binom{5}{5} = 5 - 10 + 10 - 5 + 1 = 1$

This means that if an item belongs to exactly 5 sets, it also gets counted correctly.

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose x belongs to m sets. Then the number of times x will be counted is:

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \dots + \binom{m}{m} (-1)^{m-1}$$

To prove the PIE in general, we have to show that this sum equals 1 for all m .

- Let's see a VERY clever way to prove this using the binomial theorem.

$$(\textcolor{blue}{a} + \textcolor{blue}{b})^n = \sum_{k=0}^n \binom{n}{k} \textcolor{blue}{a}^{n-k} \textcolor{blue}{b}^k \quad \text{Seems unrelated.}$$

- This is well beyond the scope of what we would expect you to figure out on your own.

Side-Quest: The Binomial Theorem for $a = 1$, $b = -1$

Suppose we plug in $a = 1$, $b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Side-Quest: The Binomial Theorem for $a = 1, b = -1$

Suppose we plug in $a = 1, b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0 if we plug in $a = 1, b = -1$:

$$0 = \sum_{k=0}^n \binom{n}{k} (-1)^k = \binom{n}{0} (-1)^0 + \binom{n}{1} (-1)^1 + \binom{n}{2} (-1)^2 + \cdots + \binom{n}{n} (-1)^n$$

Let's throw away even powers of -1 and convert odd powers of -1 to minus signs.

Side-Quest: The Binomial Theorem for $a = 1$, $b = -1$

Suppose we plug in $a = 1$, $b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$0 = \sum_{k=0}^n \binom{n}{k} (-1)^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n} (-1)^n$$

Didn't throw away or convert last one
because we don't know if even or odd.

Side-Quest: The Binomial Theorem for $a = 1$, $b = -1$

Suppose we plug in $a = 1$, $b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n} (-1)^n$$

Side-Quest: The Binomial Theorem for $a = 1, b = -1$

Suppose we plug in $a = 1, b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n} (-1)^n$$

We're close. What is $\binom{n}{0}$?

Side-Quest: The Binomial Theorem for $a = 1, b = -1$

Suppose we plug in $a = 1, b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n} (-1)^n$$

We're close. What is $\binom{n}{0}$? 1, there is one way to choose nothing.

Side-Quest: The Binomial Theorem for $a = 1$, $b = -1$

Suppose we plug in $a = 1$, $b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$0 = 1 - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n} (-1)^n$$

We're close. What is $\binom{n}{0}$? 1, there is one way to choose nothing.

Side-Quest: The Binomial Theorem for $a = 1, b = -1$

Suppose we plug in $a = 1, b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$0 = 1 - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n} (-1)^n$$

Now add $\binom{n}{1}$ to both sides.

Side-Quest: The Binomial Theorem for $a = 1, b = -1$

Suppose we plug in $a = 1, b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$\binom{n}{1} = 1 + \binom{n}{2} - \cdots + \binom{n}{n} (-1)^n$$

Then subtract $\binom{n}{2}$ from both sides.

Side-Quest: The Binomial Theorem for $a = 1, b = -1$

Suppose we plug in $a = 1, b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$\binom{n}{1} - \binom{n}{2} = 1 - \dots + \binom{n}{n} (-1)^n$$

Keep doing this, and in the last step we'll subtract $\binom{n}{n} (-1)^n$ from both sides.

Side-Quest: The Binomial Theorem for $a = 1$, $b = -1$

Suppose we plug in $a = 1$, $b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

That means the RHS is also 0:

$$\binom{n}{1} - \binom{n}{2} + \cdots - \binom{n}{n} (-1)^n = 1$$

Then, let's make that last term into a $+$ instead of a $-$ by changing the power of -1

Side-Quest: The Binomial Theorem for $a = 1, b = -1$

Suppose we plug in $a = 1, b = -1$ into the binomial theorem.

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

What number does the binomial theorem yield? LHS gives us 0 trivially.

$$(a + b)^n = (1 - 1)^n = 0^n = 0$$

After some manipulation, we have that:

$$\binom{n}{1} - \binom{n}{2} + \cdots + \binom{n}{n} (-1)^{n-1} = 1$$

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Suppose x belongs to m sets. Then the number of times x will be counted is:

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \dots + \binom{m}{m} (-1)^{m-1}$$

Wanted to show that above equals 1. On our side quest, we just showed that

$$\binom{n}{1} - \binom{n}{2} + \dots + \binom{n}{n} (-1)^{n-1} = 1$$

Thus our proof is complete since it doesn't matter if we use the letter m or n .

Quick Note #1 on Notation

The notes for this lecture use more compact notation than lecture. Alternate proof:

Rather than trying to show $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \cdots + \binom{m}{m}(-1)^{m-1} = 1$, you can use more compact sigma notation:

$$\sum_{k=1}^m (-1)^{k-1} \binom{m}{k} = 1$$

Then you can go on a side-quest with the sigma notation version of the [binomial theorem](#) to show that:

$$1 = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k}$$

Quick Note #1 on Notation (Side-Quest)

The side-quest in the notes is deriving a corollary of the binomial theorem. They do:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k = 0 = \sum_{k=0}^n (-1)^k \binom{n}{k}$$

The notes skim over the (non-trivial) fact that:

$$0 = \sum_{k=0}^n (-1)^k \binom{n}{k} \quad \rightarrow \quad 1 = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k}$$

You can show this if you'd like, I won't today (the idea is similar to what we did in lecture with expanded sum). But when the notes say "the second equality follows from" that corollary, they rely on this fact.

Quick Note #2 on Notation

In lecture I used: $|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(A_1, \dots, A_n)$

The notes use: $|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{S \subseteq \{1, \dots, n\}: |S|=k} |\cap_{i \in S} A_i|$

These two notations represent precisely the same mathematical object.

Quick Note #3 on Notation

In lecture, I walked through an example to develop intuition that let us state confidently that an item that appears in m sets is counted $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \cdots + \binom{m}{m}(-1)^{m-1}$ times, or equivalently $\sum_{k=1}^m (-1)^{k-1} \binom{m}{k}$.

The notes instead provide a compact and ultra-precise argument given below:

Proof. First, note that if $a \notin A_1 \cup \cdots \cup A_n$, then $a \notin A_i$ for all $i \in \{1, \dots, n\}$ and hence $a \notin \cap_{i \in S} A_i$ for any $S \subseteq \{1, \dots, n\}$, so it is not counted by the right-hand side (RHS) of (5). We now show that if $a \in A_1 \cup \cdots \cup A_n$, then it is counted exactly once by the RHS. Define the index set $M \subseteq \{1, \dots, n\}$ as

$$M = \{i \in \{1, \dots, n\} \mid a \in A_i\};$$

that is, $a \in A_i$ if and only if $i \in M$. Note that $m := |M| \geq 1$, since $a \in A_1 \cup \cdots \cup A_n$. Clearly, if $S \not\subseteq M$, then $a \notin \cap_{i \in S} A_i$ (i.e., if S contains an index i not in M , then $a \notin A_i$, so $a \notin \cap_{i \in S} A_i$). Furthermore, $a \in \cap_{i \in S} A_i$ for any non-empty subset $S \subseteq M$. Therefore, the number of times that a is counted on the RHS of (5) is

$$\sum_{k=1}^m (-1)^{k-1} \sum_{S \subseteq M: |S|=k} 1 = \sum_{k=1}^m (-1)^{k-1} \binom{m}{k}$$

Quick Note #3 on Notation

As you study more advanced mathematics, you will encounter dense material like this.

- Pros: Extremely compact and precise.
- Cons: Can require significant effort to unpack, potentially beyond your capability to even find a toehold.

Proof. First, note that if $a \notin A_1 \cup \dots \cup A_n$, then $a \notin A_i$ for all $i \in \{1, \dots, n\}$ and hence $a \notin \cap_{i \in S} A_i$ for any $S \subseteq \{1, \dots, n\}$, so it is not counted by the right-hand side (RHS) of (5). We now show that if $a \in A_1 \cup \dots \cup A_n$, then it is counted exactly once by the RHS. Define the index set $M \subseteq \{1, \dots, n\}$ as

$$M = \{i \in \{1, \dots, n\} \mid a \in A_i\};$$

that is, $a \in A_i$ if and only if $i \in M$. Note that $m := |M| \geq 1$, since $a \in A_1 \cup \dots \cup A_n$. Clearly, if $S \not\subseteq M$, then $a \notin \cap_{i \in S} A_i$ (i.e., if S contains an index i not in M , then $a \notin A_i$, so $a \notin \cap_{i \in S} A_i$). Furthermore, $a \in \cap_{i \in S} A_i$ for any non-empty subset $S \subseteq M$. Therefore, the number of times that a is counted on the RHS of (5) is

$$\sum_{k=1}^m (-1)^{k-1} \sum_{S \subseteq M: |S|=k} 1 = \sum_{k=1}^m (-1)^{k-1} \binom{m}{k}$$

Quick Note #3 on Notation

When I read the notes in this class, I'll often unpack them by working out small examples.

- If you're not sure, you can even ask an LLM to give some examples for you.
- "I'm looking at these notes <https://www.eecs70.org/assets/pdf/notes/n18.pdf>, and I'm confused about the proof for Theorem 18.1. Can you give me a worked out example for some specific a and four sets A_1 through A_4 ? What is the index set?"

Proof. First, note that if $a \notin A_1 \cup \dots \cup A_n$, then $a \notin A_i$ for all $i \in \{1, \dots, n\}$ and hence $a \notin \cap_{i \in S} A_i$ for any $S \subseteq \{1, \dots, n\}$, so it is not counted by the right-hand side (RHS) of (1). We now show that if $a \in A_1 \cup \dots \cup A_n$, then it is counted exactly once by the RHS. Define the index set $M \subseteq \{1, \dots, n\}$ as

$$M = \{i \in \{1, \dots, n\} \mid a \in A_i\};$$

that is, $a \in A_i$ if and only if $i \in M$. Note that $m := |M| \geq 1$, since $a \in A_1 \cup \dots \cup A_n$. Clearly, if $S \not\subseteq M$, then $a \notin \cap_{i \in S} A_i$ (i.e., if S contains an index i not in M , then $a \notin A_i$, so $a \notin \cap_{i \in S} A_i$). Furthermore, $a \in \cap_{i \in S} A_i$ for any non-empty subset $S \subseteq M$. Therefore, the number of times that a is counted on the RHS of (1) is

$$\sum_{k=1}^m (-1)^{k-1} \sum_{S \subseteq M: |S|=k} 1 = \sum_{k=1}^m (-1)^{k-1} \binom{m}{k} = 1$$

Independence

Lecture 18, CS70 Fall 2026

The Principle of Inclusion-Exclusion

Combinations of Events

- **Independence**
- Intersections and The Product Rule
- Product Rule Applications and Proof
- Unions of Events
- Unions of Events (Large N)
- Summary

Independence

Earlier, we talked about how the probability of two events is sometimes the product of their individual probabilities.

For example, consider rolling a four-sided die twice, where the probability of a 1 is $5/8$.

- Here, A_1 is the event that the first die is a 1, and we know that $P(A_1) = 5/8$.
- A_2 is the event that the second die is a 1, and we know that $P(A_2) = 5/8$.
- We said that the probability of both events happening was
$$P(A_1 \cap A_2) = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}.$$

What is notable about these two events is that they are independent!

- The outcome of each roll does not affect the other.

Independence More Formally

Two events A and B are said to be independent if:

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) \times \mathbf{P}(B)$$

Let's consider what this implies about $\mathbf{P}(A \mid B)$.

Independence More Formally

Two events A and B are said to be independent if:

$$P(A \cap B) = P(A) \times P(B)$$

Let's consider what this implies about $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \times P(B)}{P(B)} = P(A)$$

That is, knowing B tells us literally nothing new about A .

- Example: We roll 2 six-sided dice. A is the event where the dice sum to 7.
 B is the event where the first die is a 1.

Mutual Independence

We can also define the **mutual** independence of more than 2 events. Let's start with 3 events: A_1 , A_2 , and A_3 .

These 3 events are **mutually independent** if all of the following are true:

Coin Flipping

Example: Imagine we flip a fair coin twice. Define the events below:

- A_1 : First flip is heads.
- A_2 : Second flip is heads.
- A_3 : Both flips are the same.

Questions:

- Are these events pairwise independent (satisfy top three constraints below)?
- Are they mutually independent (satisfy all four constraints below)?

$$\begin{aligned}P(A_1 \cap A_2) &= P(A_1) \times P(A_2) \\P(A_1 \cap A_3) &= P(A_1) \times P(A_3) \\P(A_2 \cap A_3) &= P(A_2) \times P(A_3) \\P(A_1 \cap A_2 \cap A_3) &= P(A_1) \times P(A_2) \times P(A_3)\end{aligned}$$

<https://joshh.ug/D>



Coin Flipping

Example: Imagine we flip a fair coin twice. Define the events below:

- A_1 : First flip is heads.
- A_2 : Second flip is heads.
- A_3 : Both flips are the same.

Questions:

- Are these events pairwise independent (satisfy top three constraints)? Yes!
 - Knowing first flip is heads says nothing about second being heads (and vice versa).
 - Knowing first flip heads says nothing about both flips being same (and v-v).
 - Knowing second flip heads says nothing about both flips being same (and v-v).
- Are they mutually independent (satisfy all four constraints)?

Coin Flipping

Example: Imagine we flip a fair coin twice. Define the events below:

- A_1 : First flip is heads.
- A_2 : Second flip is heads.
- A_3 : Both flips are the same.

Questions:

- Are these events pairwise independent (satisfy top three constraints)? Yes!
- Are they mutually independent (satisfy all four constraints)?
 - No. For example, if you know first and second flips are heads, you know that both flips are the same. That is $\mathbf{P}(A_3 \mid A_1 \cap A_2) = 1$

Generalizing to 4 Events

We can expand our definition of mutually independent. For four events, we'd have 11 constraints: Six pairwise, four three-way, one four-way.

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1) \times P(A_3)$$

$$P(A_1 \cap A_4) = P(A_1) \times P(A_4)$$

$$P(A_2 \cap A_3) = P(A_2) \times P(A_3)$$

$$P(A_2 \cap A_4) = P(A_2) \times P(A_4)$$

$$P(A_3 \cap A_4) = P(A_3) \times P(A_4)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$

$$P(A_1 \cap A_2 \cap A_4) = P(A_1) \times P(A_2) \times P(A_4)$$

$$P(A_1 \cap A_3 \cap A_4) = P(A_1) \times P(A_3) \times P(A_4)$$

$$P(A_2 \cap A_3 \cap A_4) = P(A_2) \times P(A_3) \times P(A_4)$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1) \times P(A_2) \times P(A_3) \times P(A_4)$$

Generalizing to N Events

We can expand our definition of mutually independent. For n events, we have $2^n - n - 1$ constraints: $\binom{n}{2}$ pairwise, $\binom{n}{3}$ three-way, $\binom{n}{4}$ four-way,...

Can represent with compact notation that is a bit awkward (Definition 18.2 from the notes). For every subset I of $\{1, \dots, n\}$ with size $|I| \geq 2$:

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$

The notes give an even more awkward definition (Definition 18.3) in terms of each event and its complement. We won't cover this. It represents the **exact** same idea.

Intersections and The Product Rule

Lecture 18, CS70 Fall 2026

The Principle of Inclusion-Exclusion

Combinations of Events

- Independence
- **Intersections and The Product Rule**
- Product Rule Applications and Proof
- Unions of Events
- Unions of Events (Large N)
- Summary

Why Care About Intersections of Events?

In engineering, we often care about the probability of the intersection (or unions) of events. For example, if $A_1, \dots, A_n$ are all the things that have to go right, chance of success is $P(\bigcap_{i=1}^n A_i)$.



What if Events Aren't Independent?

Suppose we want to compute $P(\bigcap_{i=1}^n A_i)$, but the events are not independent?

- We can't just multiply the probabilities together.
- Example: A_1 : Both heads, A_2 : First coin heads.

What if Events Aren't Independent?

Suppose we want to compute $\mathbf{P}(\bigcap_{i=1}^n A_i)$, but the events are not independent?

- We can't just multiply the probabilities together.
- Example: A_1 : Both heads, A_2 : First coin heads. $\mathbf{P}(A_1 \cap A_2) \neq 1/4 \times 1/2$

If there are only two events, we can take advantage of our definition of conditional probability:

$$\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1)$$

Coin example above:

- $\mathbf{P}(A_1 \cap A_2) = 1/4 \times 1$
- This is because $\mathbf{P}(A_1) = 1/4$, and $\mathbf{P}(A_2 \mid A_1) = 1$.

What if Events Aren't Independent?

Suppose we want to compute $\mathbf{P} \left(\bigcap_{i \in I} A_i \right)$, but the events are not independent?

- For two events: $\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1)$

For 3 events A_1, A_2, A_3 , we have:

What if Events Aren't Independent?

Suppose we want to compute $\mathbf{P}(\bigcap_{i \in I} A_i)$, but the events are not independent?

- For two events: $\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1)$

For 3 events A_1, A_2, A_3 , we have:

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2)$$

Proof:

$$\begin{aligned} &= \mathbf{P}(A_1 \cap A_2) \times \mathbf{P}(A_3 \mid A_1 \cap A_2) \\ &= \mathbf{P}(A_1 \cap A_2 \cap A_3) \end{aligned}$$

What if Events Aren't Independent?

Suppose we want to compute $\mathbf{P}(\bigcap_{i \in I} A_i)$, but the events are not independent?

- For two events: $\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1)$
- For three events: $\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2)$

For n events $A_1, \dots, A_n$, we have the **Product Rule**:

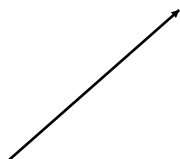
What if Events Aren't Independent?

Suppose we want to compute $\mathbf{P}(\bigcap_{i \in I} A_i)$, but the events are not independent?

- For two events: $\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1)$
- For three events: $\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2)$

For n events $A_1, \dots, A_n$, we have the **Product Rule**:

$$\mathbf{P}(\bigcap_{i=1}^n A_i) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2) \times \mathbf{P}(A_4 \mid A_1 \cap A_2 \cap A_3) \dots$$



Probability that event 4 happens given that events 1 through 3 happen.

What if Events Aren't Independent?

Suppose we want to compute $\mathbf{P} \left(\bigcap_{i \in I} A_i \right)$, but the events are not independent?

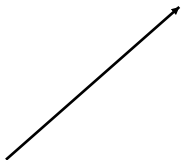
For n events $A_1, \dots, A_n$, we have the **Product Rule**:

What if Events Aren't Independent?

Suppose we want to compute $\mathbf{P}(\bigcap_{i \in I} A_i)$, but the events are not independent?

For n events $A_1, \dots, A_n$, we have the **Product Rule**:

$$\mathbf{P}(\bigcap_{i=1}^n A_i) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2) \times \cdots \times \mathbf{P}(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i) \times \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i)$$



Probability that event n happens given that events 1 through $n-1$ happen.

If we keep going, we get the expression above.

- Will prove this is true shortly.
- First let's see some applications.

Product Rule Applications and Proof

Lecture 18, CS70 Fall 2026

The Principle of Inclusion-Exclusion

Combinations of Events

- Independence
- Intersections and The Product Rule
- **Product Rule Applications and Proof**
- Unions of Events
- Unions of Events (Large N)
- Summary

Example One: Chance of a Flush in Poker

The product rule makes some probability calculations easier. Consider trying to calculate the probability of a heart flush. Recall that a flush is all 5 cards having the same suit.

Example Flushes: {A♥, 2♥, 5♥, K♥, Q♥} {5♥, A♥, 2♥, Q♥, K♥}

Using our new framework, if A is the event where we get a heart flush, and A_i is the event where the i th draw is a heart:

$$P(A) = P\left(\bigcap_{i=1}^5 A_i\right)$$

Example One: Chance of a Flush in Poker

If A is the event where we get a heart flush, and A_i is the event where the i th draw is a heart:

$$P(A) = P\left(\bigcap_{i=1}^5 A_i\right)$$

We can write this using the **product rule** as:

$$P(A) = P(A_1) \times P(A_2 \mid A_1) \times P(A_3 \mid A_1 \cap A_2) \times \cdots \times P\left(A_5 \mid \bigcap_{i=1}^4 A_i\right)$$

Example Flushes: {A♥, 2♥, 5♥, K♥, Q♥}

{5♥, A♥, 2♥, Q♥, K♥}

Example One: Chance of a Flush in Poker

If A is the event where we get a heart flush, and A_i is the event where the i th draw is a heart:

$$P(A) = P\left(\bigcap_{i=1}^5 A_i\right)$$

We can write this using the **product rule** as:

$$P(A) = P(A_1) \times P(A_2 \mid A_1) \times P(A_3 \mid A_1 \cap A_2) \times \cdots \times P\left(A_5 \mid \bigcap_{i=1}^4 A_i\right)$$

$$P(A) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} \approx 0.0005$$

Easier than counting number of flushes and dividing by number of hands!

Example Flushes: {A♥, 2♥, 5♥, K♥, Q♥}

{5♥, A♥, 2♥, Q♥, K♥}

Example Two: Monty Hall

Can think of Monty Hall in terms of the product rule.

Let:

- C_i be the event that contestant chooses door i .
- P_i be the event that the prize is behind door i .
- H_i be the event that the host shows the goat behind door i .

Example Two: Monty Hall

Can think of Monty Hall in terms of the product rule.

Let:

- C_i be the event that contestant chooses door i .
- P_i be the event that the prize is behind door i .
- H_i be the event that the host shows the goat behind door i .

Example: $\mathbf{P}(C_1 \cap P_2 \cap H_3) = \mathbf{P}(C_1) \times \mathbf{P}(P_2 \mid C_1) \times \mathbf{P}(H_3 \mid C_1 \cap P_2)$

- $\mathbf{P}(C_1) = \frac{1}{3}$, $\mathbf{P}(P_2 \mid C_1) = \frac{1}{3}$, $\mathbf{P}(H_3 \mid C_1 \cap P_2) = 1$
- Thus $\mathbf{P}(C_1 \cap P_2 \cap H_3) = 1/9$

See notes for more exploration of this idea.

Example Three: Coin Tosses

As an example of a degenerate case, let's consider 3 independent coin tosses using the product rule.

Let A be the event that all three tosses are heads.

Example Three: Coin Tosses

As an example of a degenerate case, let's consider 3 independent coin tosses using the product rule.

Let A be the event that all three tosses are heads.

$$\begin{aligned} \mathbf{P}(A) &= \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_2 \cap A_1) \\ &= \mathbf{P}(A_1) \times \mathbf{P}(A_2) \times \mathbf{P}(A_3) \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{8} \end{aligned}$$

Product Rule Proof

For n events $A_1, \dots, A_n$, we have the **Product Rule**:

$$P\left(\bigcap_{i=1}^n A_i\right) = P(A_1) \times P(A_2 \mid A_1) \times P(A_3 \mid A_1 \cap A_2) \times \cdots \times P(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i) \times P(A_n \mid \bigcap_{i=1}^{n-1} A_i)$$

Proof by induction. Base Case: $n = 1$

- Trivially: $P(A_1) = P(A_1)$

Product Rule Proof

For n events $A_1, \dots, A_n$, we have the **Product Rule**:

$$\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2) \times \cdots \times \mathbf{P}(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i) \times \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i)$$

Proof by induction. Assume inductive hypothesis for $n - 1$ and prove it holds for n .

- Assume $\mathbf{P} \left(\bigcap_{i=1}^{n-1} A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \cdots \times \mathbf{P}(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i)$
- Want to show $\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \cdots \times \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i)$
- We know $\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P} \left(A_n \cap \left(\bigcap_{i=1}^{n-1} A_i \right) \right)$
 - This is just a simple restatement!
 - Similar to saying $3 + 4 + 5 + 6 = 3 + (4 + 5 + 6)$

Product Rule Proof

For n events $A_1, \dots, A_n$, we have the **Product Rule**:

$$\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2) \times \cdots \times \mathbf{P}(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i) \times \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i)$$

Proof by induction. Assume inductive hypothesis for $n - 1$ and prove it holds for n .

- Assume $\mathbf{P} \left(\bigcap_{i=1}^{n-1} A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \cdots \times \mathbf{P}(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i)$
- Want to show $\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \cdots \times \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i)$
- We know $\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P} \left(A_n \cap \left(\bigcap_{i=1}^{n-1} A_i \right) \right)$

$$= \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i) \times \mathbf{P} \left(\bigcap_{i=1}^{n-1} A_i \right)$$

By definition: $\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$, here $A = A_n$ and $B = \bigcap_{i=1}^{n-1} A_i$

Product Rule Proof

For n events $A_1, \dots, A_n$, we have the **Product Rule**:

$$\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2) \times \cdots \times \mathbf{P}(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i) \times \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i)$$

Proof by induction. Assume inductive hypothesis for $n - 1$ and prove it holds for n .

- Assume $\mathbf{P} \left(\bigcap_{i=1}^{n-1} A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \cdots \times \mathbf{P}(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i)$
- Want to show $\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \cdots \times \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i)$
- We know $\mathbf{P} \left(\bigcap_{i=1}^n A_i \right) = \mathbf{P} \left(A_n \cap \left(\bigcap_{i=1}^{n-1} A_i \right) \right)$

$$= \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i) \times \mathbf{P} \left(\bigcap_{i=1}^{n-1} A_i \right)$$

$$= \mathbf{P}(A_n \mid \bigcap_{i=1}^{n-1} A_i) \times \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \cdots \times \mathbf{P}(A_{n-1} \mid \bigcap_{i=1}^{n-2} A_i)$$

QED

Unions of Events

Lecture 18, CS70 Fall 2026

The Principle of Inclusion-Exclusion

Combinations of Events

- Independence
- Intersections and The Product Rule
- Product Rule Applications and Proof
- **Unions of Events**
- Unions of Events (Large N)
- Summary

The Three Four-Sided Dice Game

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.

Claim: The probability of winning is 75%

- $\mathbf{P}(A) = \mathbf{P}(A_1 \cup A_2 \cup A_3) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) = 3 \times \frac{1}{4} = \frac{3}{4}$

Is this claim correct?

The Three Four-Sided Dice Game

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.

Claim: The probability of winning is 75%

- $P(A) = P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) = 3 \times \frac{1}{4} = \frac{3}{4}$

Is this claim correct? No!

- Imagine we roll four fair four-sided dice.
- This argument would imply the chance of winning is $4 \times \frac{1}{4} = 1$
- Or that the probability of winning with 6 dice is 1.5.

Overlapping Events

The issue is that the events are not disjoint.

- Two events are **disjoint** if they share no sample points.

Consider the sample space for three four-sided die rolls:

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

Overlapping Events

The issue is that the events are not disjoint.

- Two events are **disjoint** if they share no sample points.

Consider the sample space for three four-sided die rolls:

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

A_1 is the event where the first die is 4.

Overlapping Events

The issue is that the events are not disjoint.

- Two events are **disjoint** if they share no sample points.

Consider the sample space for three four-sided die rolls:

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

$$P(A_1 \cup A_2) \neq P(A_1) + P(A_2)$$

Some points are in both events, so adding probabilities double counts some of the points!

A_1 is the event where the first die is 4.

A_2 is the event where the second die is 4.

Solution: Use the Inclusion-Exclusion Formula

We'll need to use the inclusion-exclusion formula from the start of lecture.

Reminder, this formula told us about the size of the union of arbitrary subsets of the same finite set A . Specifically:

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{S \subseteq \{1, \dots, n\}: |S|=k} \left| \bigcap_{i \in S} A_i \right|$$

Above, I'm using the notation from the notes. This notation is different (but identical in meaning) to the notation from earlier today:

$$|\textcolor{red}{A}_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \text{INTERSECTSUM}_k(\textcolor{red}{A}_1, \dots, A_n)$$

Using the Principle of Inclusion-Exclusion in the Context of Probability

The PIE formula lets us count the size of the union of several potentially overlapping subsets:

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \sum_{S \subseteq \{1, \dots, n\}: |S|=k} \left| \bigcap_{i \in S} A_i \right|$$

Naturally, we can use this to compute the probability of the union of events.

$$\mathbf{P}(A_1 \cup \dots \cup A_n) = \sum_{k=1}^n (-1)^{k-1} \sum_{S \subseteq \{1, \dots, n\}: |S|=k} \mathbf{P} \left(\bigcap_{i \in S} A_i \right)$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- Let A_i be the event that we get a 4 on the i th roll.

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\ &\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\ &\quad + P(A_1 \cap A_2 \cap A_3) \end{aligned}$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- Let A_i be the event that we get a 4 on the i th roll.
- What is $\mathbf{P}(A_1 \cap A_2)$?

$$\mathbf{P}(A_1 \cup A_2 \cup A_3) = 1/4 + 1/4 + 1/4$$

$$\begin{aligned} & - \mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1 \cap A_3) - \mathbf{P}(A_2 \cap A_3) \\ & + \mathbf{P}(A_1 \cap A_2 \cap A_3) \end{aligned}$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- Let A_i be the event that we get a 4 on the i th roll.
- What is $\mathbf{P}(A_1 \cap A_2)$? Probability of getting two fours on **independent** die rolls, so $1/4 \times 1/4 = 1/16$

$$\mathbf{P}(A_1 \cup A_2 \cup A_3) = 1/4 + 1/4 + 1/4$$

$$- 1/16 - 1/16 - 1/16$$

$$+ \mathbf{P}(A_1 \cap A_2 \cap A_3)$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- Let A_i be the event that we get a 4 on the i th roll.
- What is $\mathbf{P}(A_1 \cap A_2 \cap A_3)$?

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup A_3) &= 1/4 + 1/4 + 1/4 \\ &\quad - 1/16 - 1/16 - 1/16 \\ &\quad + \mathbf{P}(A_1 \cap A_2 \cap A_3)\end{aligned}$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- Let A_i be the event that we get a 4 on the i th roll.
- What is $\mathbf{P}(A_1 \cap A_2 \cap A_3)$? $1/64$

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup A_3) &= 1/4 + 1/4 + 1/4 \\ &\quad - 1/16 - 1/16 - 1/16 \\ &\quad + 1/64\end{aligned}$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- Let A_i be the event that we get a 4 on the i th roll.
- Overall probability is: $\frac{3}{4} - \frac{3}{16} + \frac{1}{64} = \frac{48}{64} - \frac{12}{64} + \frac{1}{64} = \frac{37}{64}$

$$\begin{aligned} \mathbf{P}(A_1 \cup A_2 \cup A_3) &= 1/4 + 1/4 + 1/4 \\ &\quad - 1/16 - 1/16 - 1/16 \\ &\quad + 1/64 \end{aligned}$$

This all worked because $\mathbf{P}(\bigcap_i A_i) = \prod_i \mathbf{P}(A_i)$ for independent events, which let us easily compute values like $\mathbf{P}(A_1 \cap A_2 \cap A_3)$.

Visualization of our Usage of the PIE

Let's see a visual picture.

First, we added $P(A_1) + P(A_2) + P(A_3)$, giving us 48/64.

- However, we've double counted and triple counted some points!

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

Sample points

0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
1	1	1	2	1	1	1	2
0	0	0	1	1	1	1	2
0	0	0	1	1	1	1	2
0	0	0	1	1	1	1	2
1	1	1	2	2	2	2	3

Times each point is counted

Visualization of our Usage of the PIE

First we added $P(A_1) + P(A_2) + P(A_3)$, giving us $48/64$.

- So then we subtracted $P(A_1 \cap A_2) = 4/64$ to avoid double counting $A_1 \cap A_2$.

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	4X1	4X2	4X3	4X4

Sample points

0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
1	1	1	2	1	1	1	2
0	0	0	1	1	1	1	2
0	0	0	1	1	1	1	2
0	0	0	1	1	1	1	2
1	1	1	2	1	1	1	2

Times each point is counted

Visualization of our Usage of the PIE

First we added $P(A_1) + P(A_2) + P(A_3)$, giving us 48/64.

- So then we subtracted $P(A_1 \cap A_2) = 4/64$ to avoid double counting $A_1 \cap A_2$.
- Then we subtracted $P(A_2 \cap A_3) = 4/64$ to avoid double counting $A_2 \cap A_3$.

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

Sample points

0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	2
0	0	0	1	1	1	1	2
0	0	0	1	1	1	1	2
1	1	1	1	1	1	1	1

Times each point is counted

Note the bottom right point is now added three times and subtracted two times.

Visualization of our Usage of the PIE

First we added $P(A_1) + P(A_2) + P(A_3)$, giving us 48/64.

- So then we subtracted $P(A_1 \cap A_2) = 4/64$ to avoid double counting $A_1 \cap A_2$.
- Then we subtracted $P(A_2 \cap A_3) = 4/64$ to avoid double counting $A_2 \cap A_3$.
- Then we subtracted $P(A_1 \cap A_3) = 4/64$ to avoid double counting $A_1 \cap A_3$.

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

Sample points

0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1
0	0	0	1	1	1	1	1
0	0	0	1	1	1	1	1
1	1	1	1	1	1	1	0

Times each point is counted

Note the bottom right point is now added three times and subtracted three times.

Visualization of our Usage of the PIE

First we added $P(A_1) + P(A_2) + P(A_3)$, giving us 48/64.

- So then we subtracted $P(A_1 \cap A_2) = 4/64$ to avoid double counting $A_1 \cap A_2$.
- Then we subtracted $P(A_2 \cap A_3) = 4/64$ to avoid double counting $A_2 \cap A_3$.
- Then we subtracted $P(A_1 \cap A_3) = 4/64$ to avoid double counting $A_1 \cap A_3$.

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

Sample points

0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	1
1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1
0	0	0	1	1	1	1	1
0	0	0	1	1	1	1	1
1	1	1	1	1	1	1	1

Times each point is counted

Lastly, we add back in $P(A_1 \cap A_2 \cap A_3)$, giving us $\frac{48-4-4-4+1}{64} = \frac{37}{64}$

Unions of Events (Large N)

Lecture 18, CS70 Fall 2026

The Principle of Inclusion-Exclusion

Combinations of Events

- Independence
- Intersections and The Product Rule
- Product Rule Applications and Proof
- Unions of Events
- **Unions of Events (Large N)**
- Summary

The Principle of Inclusion-Exclusion

Mathematically, we can use PIE to compute the probability of unions.

Practically, this becomes challenging even for modestly sized values of n .

Example: Suppose we have 100 events and want to compute the probability of their union.

- Have to add (or subtract) every pairwise, three-way, four-way, etc. probability.
- There are $2^n - 1$ such terms, because $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1$. You can prove this with the binomial theorem with $a = 1$, $b = 1$ (then drop the $\binom{n}{0} = 1$ term).
- 2^{100} is too long to wait for an answer.

Inclusion-Exclusion: Mutually Exclusive Events

This isn't a problem if our events are mutually exclusive, i.e. $P(A_i \cap A_j) = 0$ for all $i \neq j$, or equivalently $A_i \cap A_j = \emptyset$ for all $i \neq j$.

- Just sum the probabilities of the individual events.

We've actually already used this fact before.

Example: Computing the probability of a flush.

- $F_{\spadesuit}$:  flush, $F_{\heartsuit}$:  flush, $F_{\diamondsuit}$:  flush, $F_{\clubsuit}$:  flush
- Flushes in different suits are clearly disjoint (i.e. mutually exclusive)!
 - $P(F_{\spadesuit} \cup F_{\heartsuit} \cup F_{\diamondsuit} \cup F_{\clubsuit}) = P(F_{\spadesuit}) + P(F_{\heartsuit}) + P(F_{\diamondsuit}) + P(F_{\clubsuit})$

Inclusion-Exclusion: Independent Events

If the events have overlap, but are mutually independent, there's another shortcut we can use: $\mathbf{P}(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - \mathbf{P}(\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_n})$

$$= 1 - \mathbf{P}(\overline{A_1}) \times \mathbf{P}(\overline{A_2}) \times \dots \times \mathbf{P}(\overline{A_n})$$

Example: Let A_i be the event that we get a 4 on the i th roll of a 4-sided die.

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup A_3) &= 1 - \mathbf{P}(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}) \\ &= 1 - \mathbf{P}(\overline{A_1})\mathbf{P}(\overline{A_2})\mathbf{P}(\overline{A_3}) \\ &= 1 - \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \\ &= 1 - \frac{27}{64} = \frac{37}{64}\end{aligned}$$

Note: This technique is not in the notes.

Inclusion-Exclusion: Approximation

If we can't use any tricks, we can also truncate the summation:

$$\mathbf{P}(A_1 \cup \dots \cup A_n) = \sum_{k=1}^n (-1)^{k-1} \sum_{S \subseteq \{1, \dots, n\}: |S|=k} \mathbf{P}\left(\bigcap_{i \in S} A_i\right)$$

Let $n_s < n$ be the maximum value of k through which we sum.

- If n_s is odd, the computed value is an overestimate.
- If n_s is even, the computed value is an underestimate.

Example: For our dice game with 3 four-sided dice:

- $n_s = 1$, $\mathbf{P}(A_1 \cup A_2 \cup A_3) \approx \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3)$ [overestimate]
- $n_s = 2$, $\mathbf{P}(A_1 \cup A_2 \cup A_3) \approx \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) - \mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_2 \cap A_3) - \mathbf{P}(A_1 \cap A_3)$ [underestimate]

Inclusion-Exclusion: Approximation

If we can't use any tricks, we can also truncate the summation:

$$\mathbf{P}(A_1 \cup \dots \cup A_n) = \sum_{k=1}^n (-1)^{k-1} \sum_{S \subseteq \{1, \dots, n\}: |S|=k} \mathbf{P}\left(\bigcap_{i \in S} A_i\right)$$

Let $n_s < n$ be the maximum value of k through which we sum.

- If n_s is odd, the computed value is an overestimate.
- If n_s is even, the computed value is an underestimate.

Inclusion-Exclusion: The Union Bound

The union bound (also called Boole's inequality) comes from a special case of our approximation, specifically the case where $n_s = 1$.

- For example, for the dice game, just summing $\mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3)$.

The union bound states:

$$\mathbf{P}(A_1 \cup \dots \cup A_n) \leq \sum_{i=1}^n \mathbf{P}(A_i)$$

Proving this is given as an exercise in the notes.

- Can also go further and prove that as you increase n_s , you switch between overestimates and underestimates.

Summary

Lecture 18, CS70 Fall 2026

The Principle of Inclusion-Exclusion

Combinations of Events

- Independence
- Intersections and The Product Rule
- Product Rule Applications and Proof
- Unions of Events
- Unions of Events (Large N)
- **Summary**

Today we covered two topics:

- The Principle of Inclusion-Exclusion (PIE) for the size of a union of sets.
- Combinations of events
 - Easy when the events are independent.
 - Intersections of non-independent events: Use product rule.
 - Unions of non-independent events: Can use the principle of inclusion-exclusion, though the number of terms blows up exponentially.
 - Unions of disjoint events: No blow-up occurs. Just add the probabilities.
 - Unions of independent events: Avoid PIE blow-up by instead computing 1 minus the probability of the intersection of the complements of the events.
 - Other unions: Truncate for an approximation, e.g. ignoring all pairwise, three-way, etc. terms. $\sum_{i=1}^n \mathbf{P}(A_i)$ is called the union bound.

Reflection on HW8

Different approaches to the same mathematical problem may be radically different in difficulty.

Example from HW8: How many ways are there to give 8 identical candies to 3 children so that every child gets at least one?

- One approach: give each child one candy first, then hand out the other 5 however you like: $\binom{7}{2} = 21$.
- Another approach: count all $\binom{10}{2} = 45$ ways, then subtract those where some child gets nothing. With $A_i = \text{"child } i \text{ gets nothing"}$, PIE gives $|A_1 \cup A_2 \cup A_3| = 3 \cdot 9 - 3 \cdot 1 + 0 = 24$, so the answer is $45 - 24 = 21$.

Both work, but the first approach is much easier. The events A_i overlap, so the second approach needs PIE.