Continuous Probability Distributions

Lecture 24, CS70 Fall 2026

Continuous Probability Distributions

The Exponential Distribution

Expectation and Variance

Joint Distributions and Independence

Summary and Reflections

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Framing: A Random Variable with Range [0, 2]

Imagine that we choose a real number X, uniformly at random from $[0, 2]$

What is the probability that $X = 0.3$? 0

What is $P(X \leq 1/2)$? $1/4 = 25\%$

What is $P(X \in [0.7, 0.9])$? $\frac{0.2}{2} = 0.1 = 10\%$

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Framing: A Random Variable with Range [0, 2]

Imagine that we choose a real number X, uniformly at random from $[0, 2]$

What is $P(X \in [a, b])$?

- Following the pattern from $P(X \in [0.7, 0.9])$, we can fill in the table below:

$P(X = 0.3)$	0
$P(X \leq 1/2)$	25%
$P(X \in [0.7, 0.9])$	10%
$P(X \in [a, b])$	$\frac{b-a}{2}$

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Review: Distributions of Discrete Random Variables

For a discrete random variable, we have the notion of the distribution of a random variable.

- Enumeration of every value and its probability.

a	$P(X = a)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

The function that maps each value to its probability is called the **Probability Mass Function**.

Examples:

- The PMF of rolling two six-sided dice is to the right.
- The PMF of $X \sim \text{Binomial}(n,\ p)$ is $\binom{n}{k} p^k (1-p)^{n-k}$
- The PMF of $X \sim \text{Geometric}(p)$ is $(1-p)^{k-1} p$

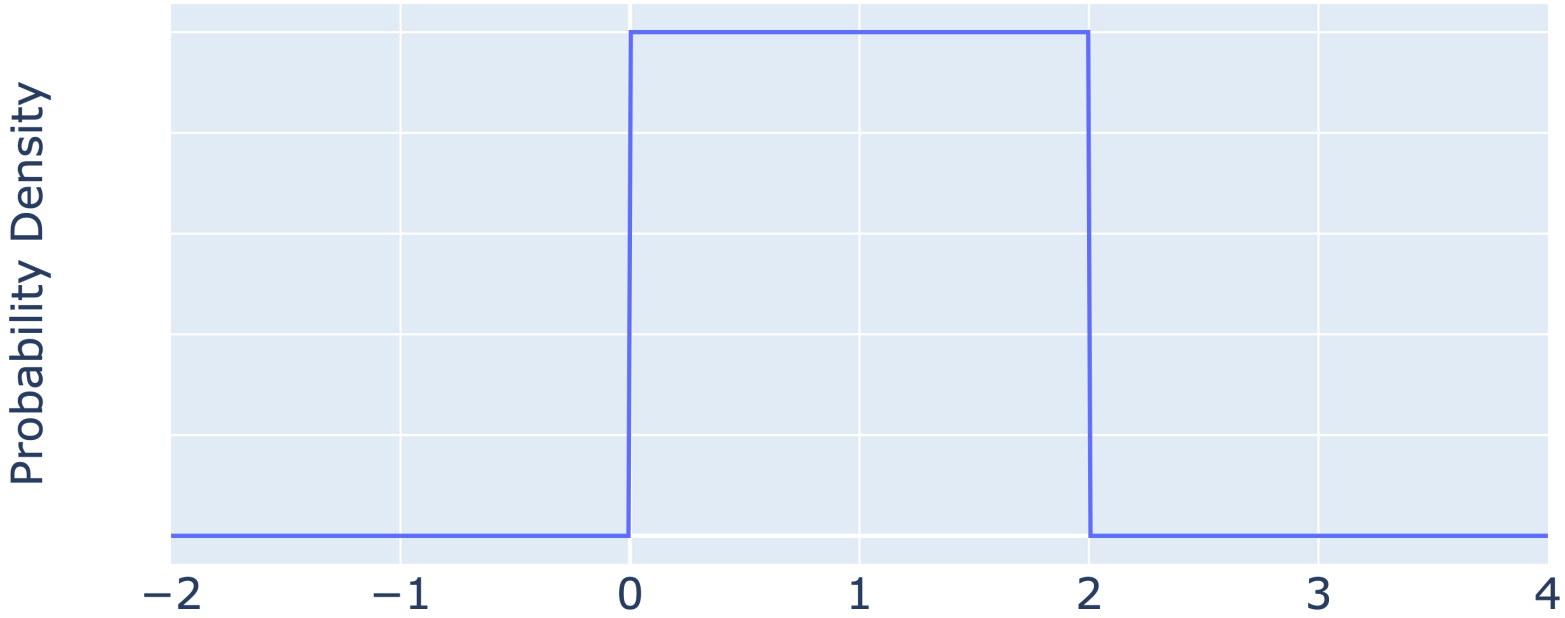
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Framing: A Random Variable with Range [0, 2]

Imagine that we choose a real number X uniformly at random from $[0, 2]$.

- For this random variable, the probability mass function is not useful. $P(X = x) = 0$ for all $x \in [0, 2]$.

There is, however, a useful analogue called the "probability density function", which looks like this:

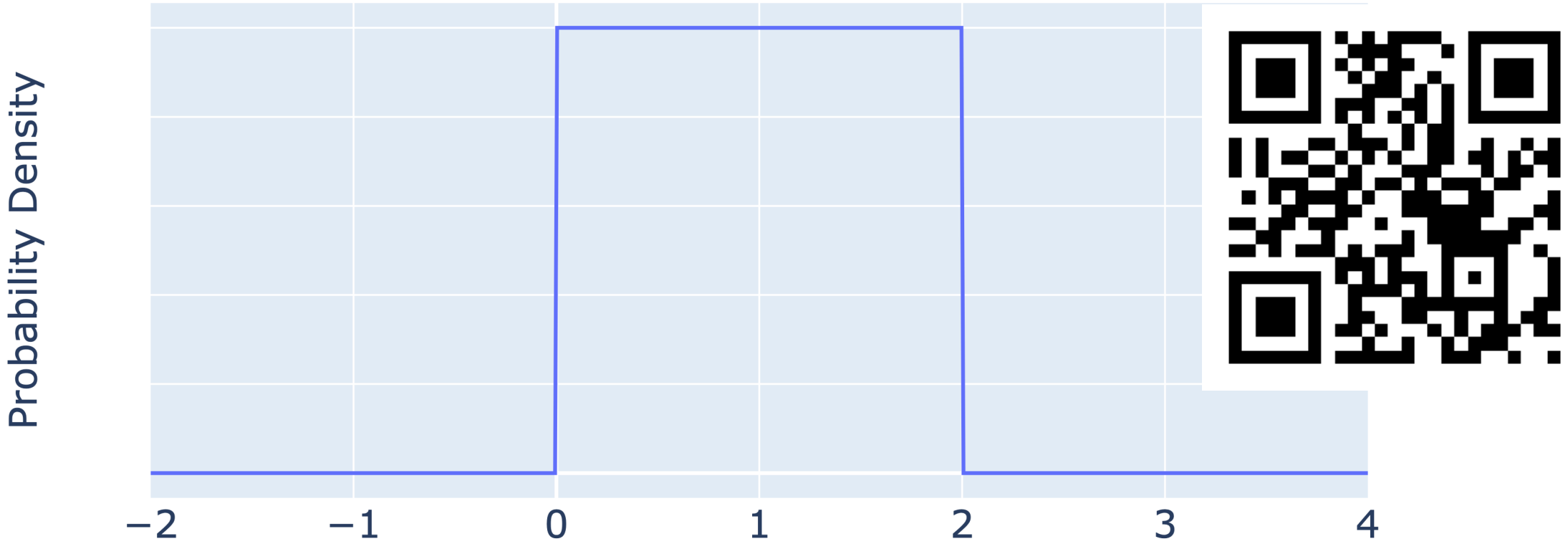


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Framing: A Random Variable with Range [0, 2]

Imagine that we choose a real number X uniformly at random from $[0, 2]$.

- The "probability mass function" is not useful, always zero.
- The "probability density function" looks like this:



What should be the y-value of the PDF in the non-zero region? https://joshh.ug/A

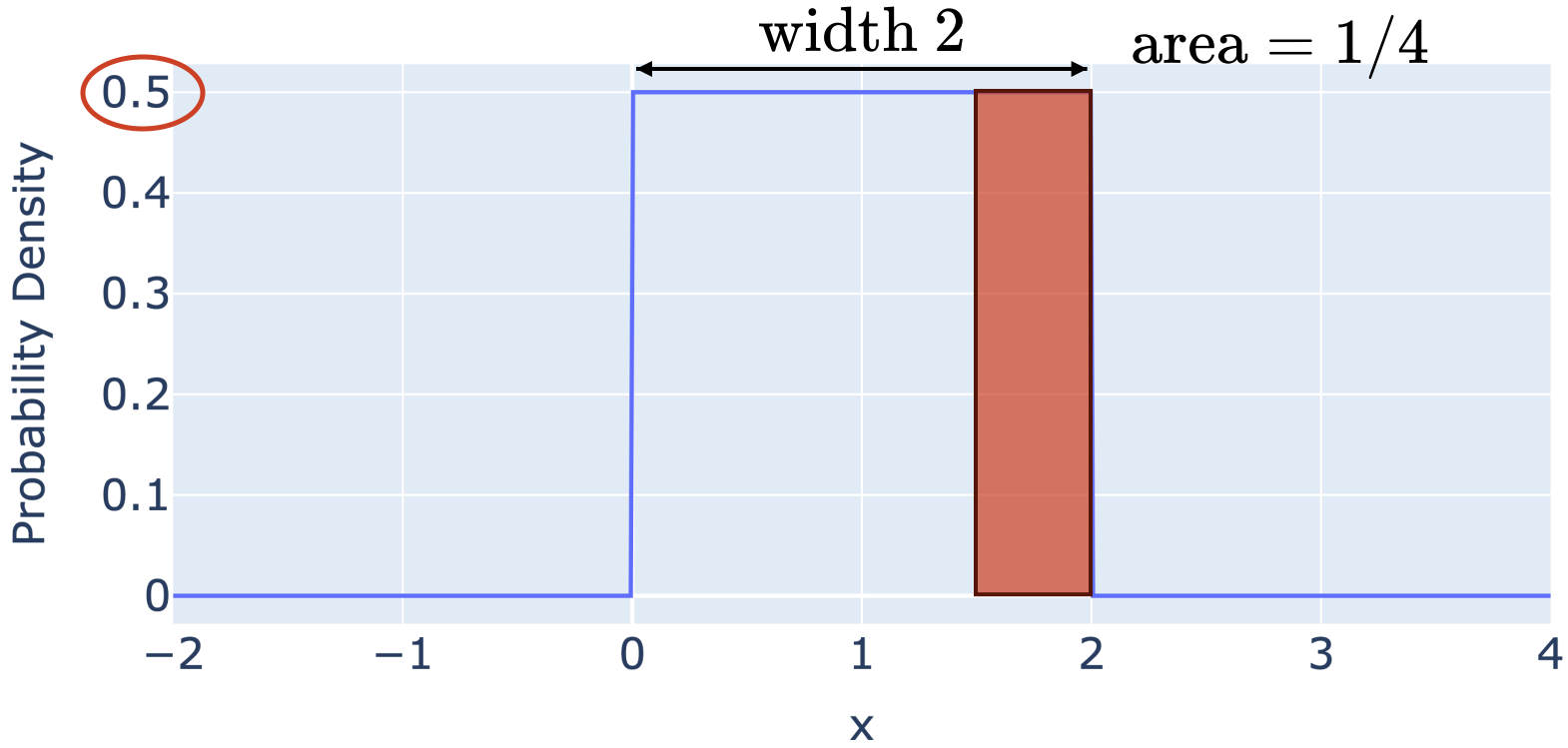
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Framing: A Random Variable with Range [0, 2]

Imagine that we choose a real number X, uniformly at random from $[0, 2]$

$P(X \in [a, b])$ = area under the PDF. Example: $[1.5, 2]$

This is a rectangle with width 2. Want area to be 1,
so height should be 0.5.

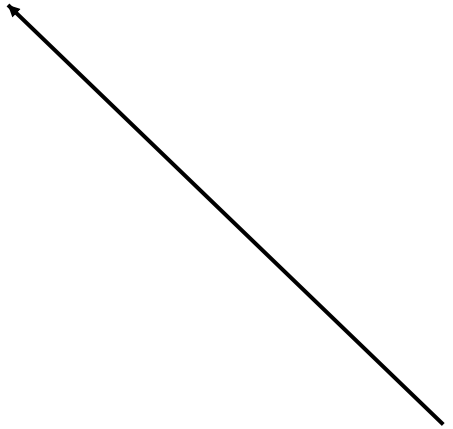


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Probability Density Function (as described in the notes)

Definition (Probability Density Function). A probability density function (PDF) for a real-valued random variable X is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

1. f is non-negative, i.e. $f(x) \geq 0 \ \ \forall x \in \mathbb{R}$
2. Total integral of f is 1, i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$



PDF is usually lowercase f

Distribution of X is given by:

$P(X \in [a,b]) = P(a \leq X \leq b) = \int_a^b f(x)\,dx$ for all $a < b$

Can think of $f(x)$ as the "probability per unit length" in the vicinity of x.

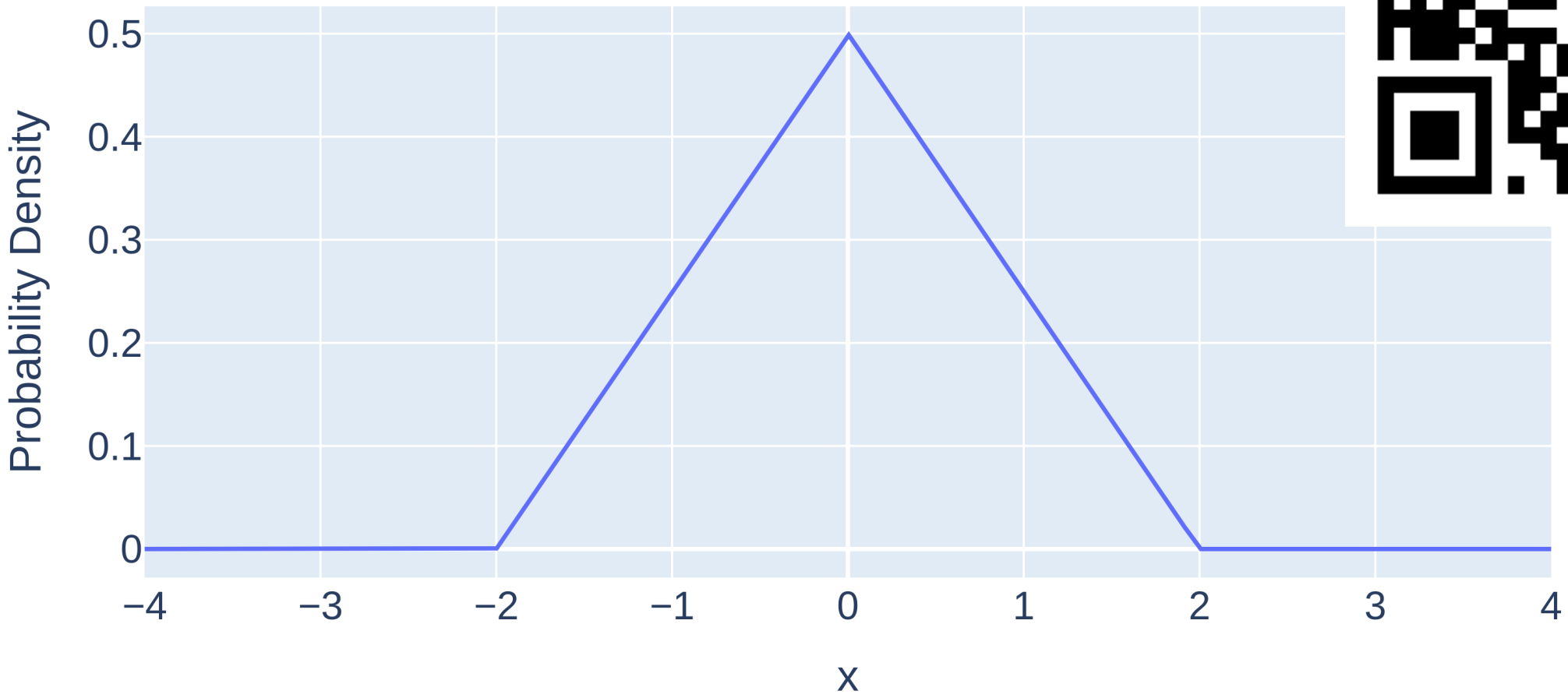
- Values of $f(x)$ are not probabilities, and may be > 1.

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Another Example Distribution: The Triangle Distribution

Consider a random variable X with the PDF below:

- What is $P(X \leq -1)$?
- https://joshh.ug/B

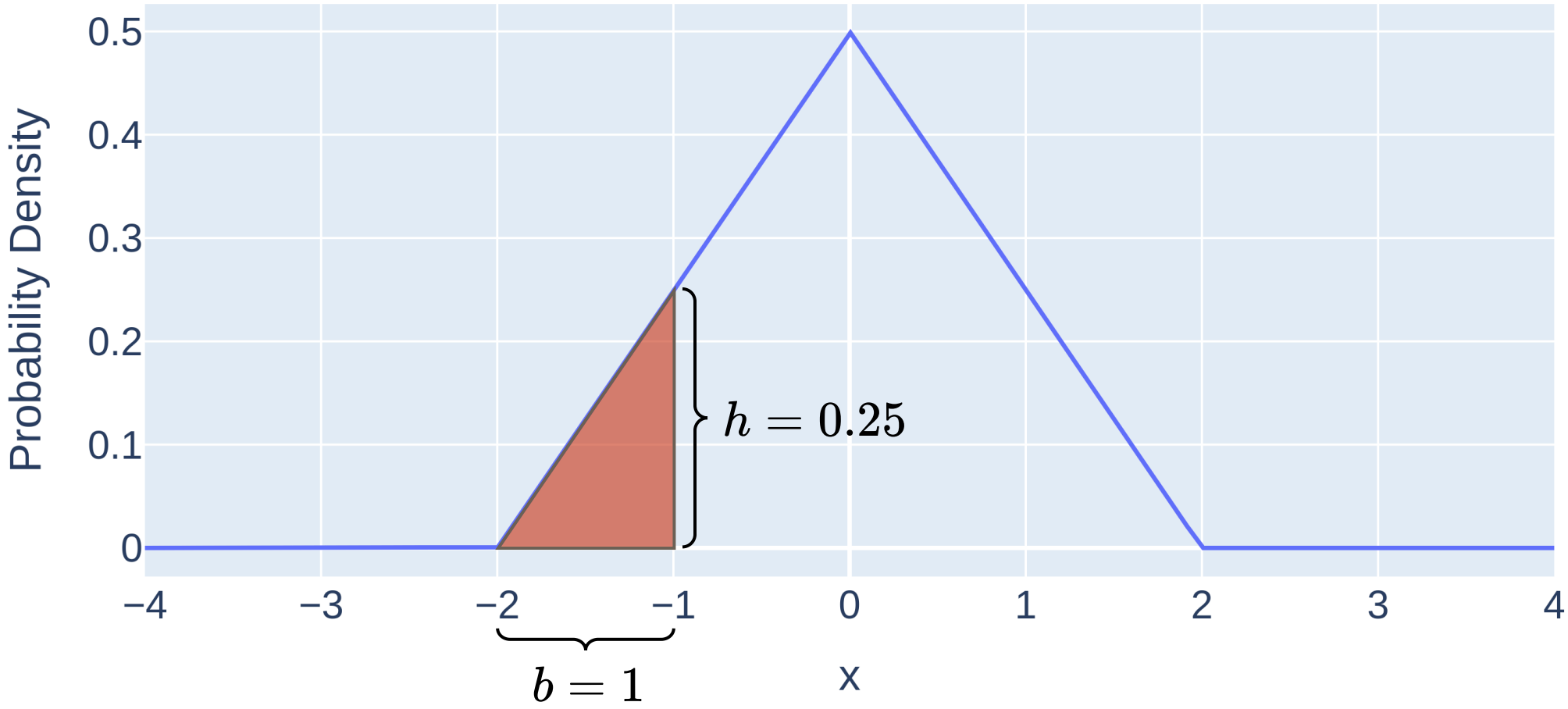


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Another Example Distribution: The Triangle Distribution

Consider a random variable X with the PDF below:

- What is $P(X \leq -1)$?
- This is a right triangle with base 1, height 1/4, so area is $bh/2 = 1/8 = 0.125$



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Another Example Distribution: The Triangle Distribution

Consider a random variable X with the PDF below:

- What is $P(X \leq -1)$? Can also compute using integration.

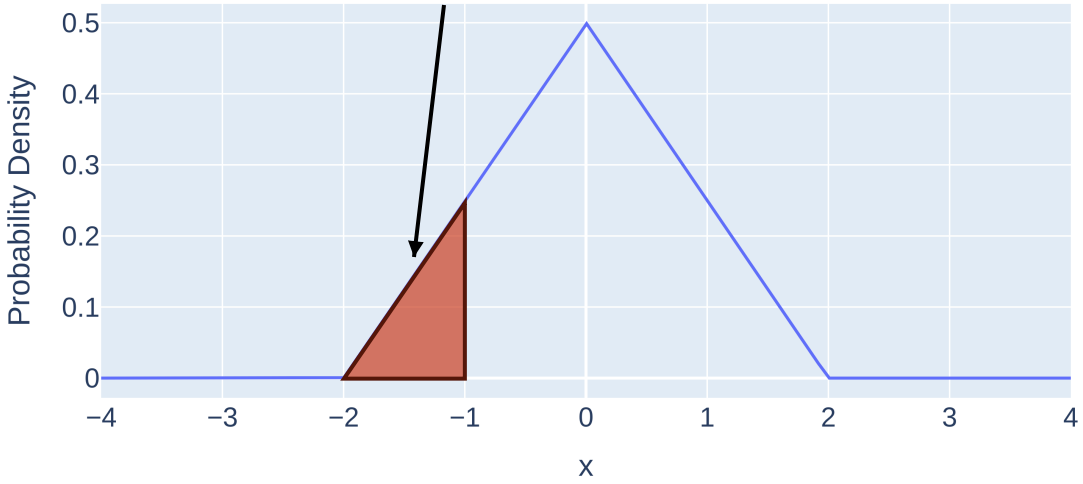
$$P(-\infty \leq X \leq -1) = \int_{-\infty}^{-1} f(x)dx = \int_{-2}^{-1} \left(\frac{1}{4}x + \frac{1}{2}\right) dx$$

$$= \frac{1}{8}x^2 + \frac{1}{2}x \Big|_{-2}^{-1}$$

$$= \left(\frac{1}{8} - \frac{1}{2}\right) - \left(\frac{1}{8} \cdot 4 - \frac{2}{2}\right)$$

$$= \frac{1}{8} - \frac{1}{2} - \frac{1}{2} + 1 = \frac{1}{8}$$

$$f(x) = mx + b = \tfrac{1}{4}x + \tfrac{1}{2}$$

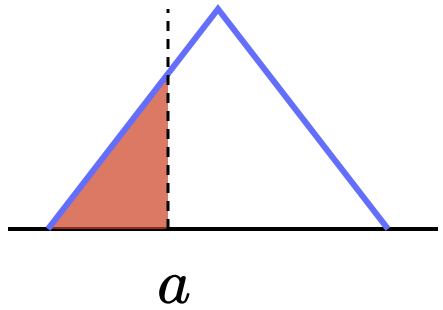


Hypotenuse of triangle is a line with slope 0.25 and y-intercept 0.5

The Cumulative Distribution Function

In the previous exercise, we computed:

$$P(-\infty \leq X \leq a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$



This has a special name: The **Cumulative Distribution Function** (CDF).

- The CDF tells us the probability that the random variable is $\leq a$.
- The CDF is the integral of the PDF from $-\infty$ to a.
- The CDF is often given as capital F, e.g. $F(a) = P(X \leq a)$.
- Likewise the PDF is the derivative with respect to a of the CDF:

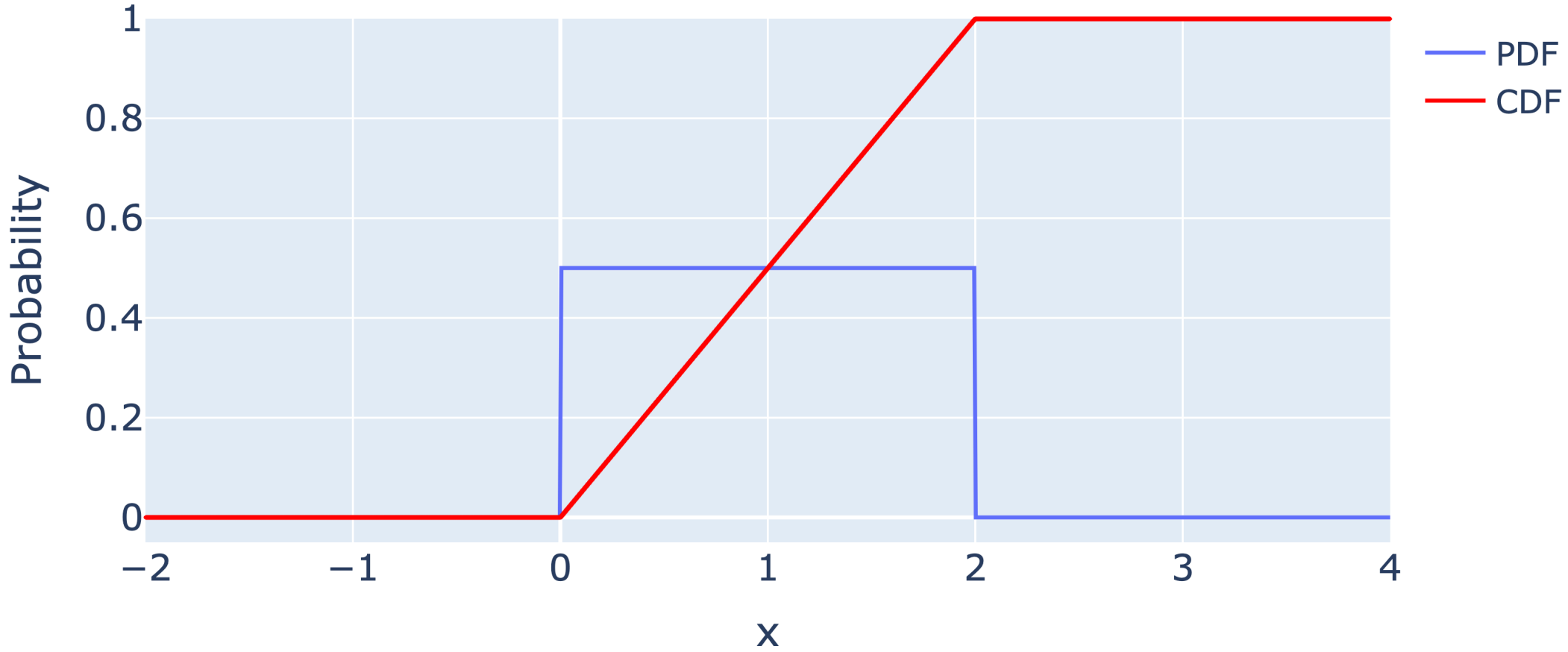
$$f(a) = \frac{dF(a)}{da}$$

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Cumulative Distribution Function Example 1: Uniform Distribution

Below, we plot the PDF and CDF of the Uniform Distribution together.

- Note: The CDF is the integral of the PDF from $-\infty$ to x.
- Note: The CDF always grows towards 1 (and never exceeds it).



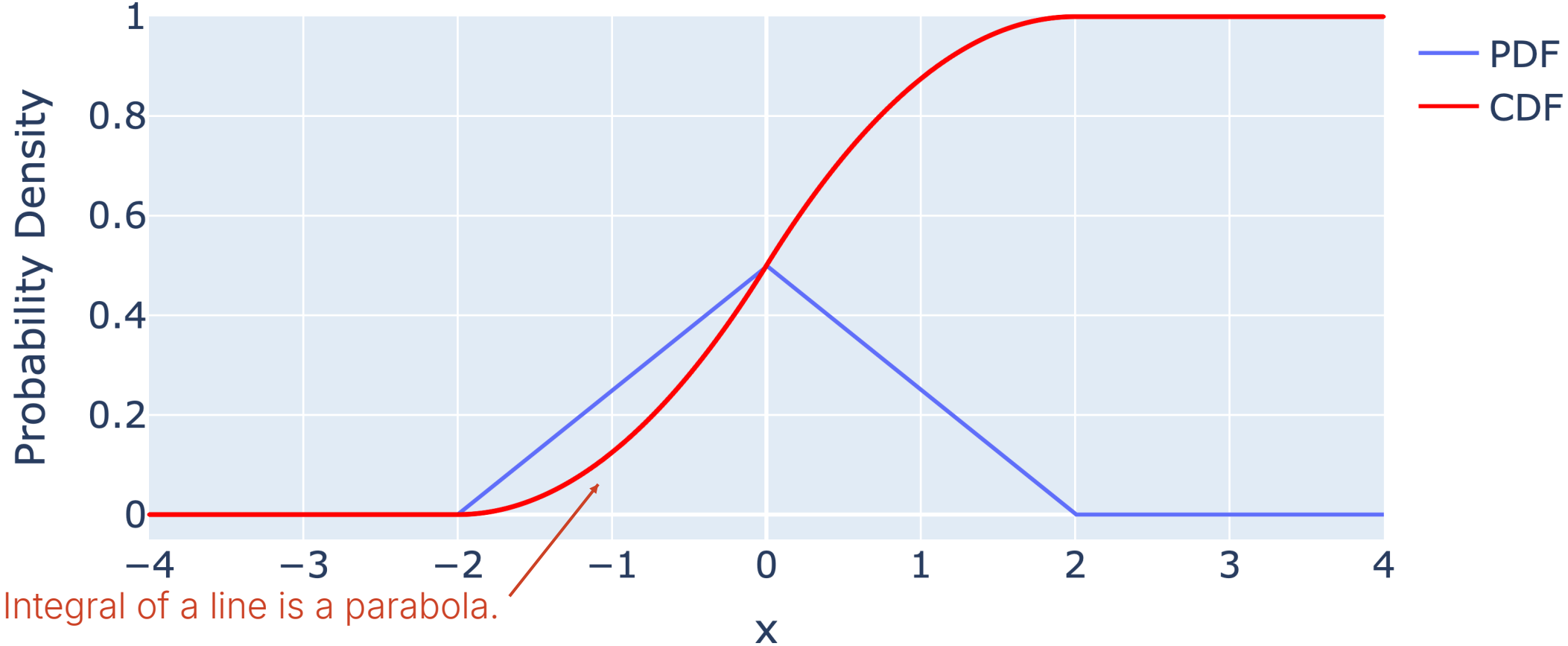
PDF: Probability per unit length

CDF: $P(X \leq x)$

Cumulative Distribution Function Example 2: Triangle Distribution

Below, we plot the PDF and CDF of the Triangle Distribution together.

- Note: The CDF is the integral of the PDF from $-\infty$ to x.
- Note: The CDF always grows towards 1 (and never exceeds it).



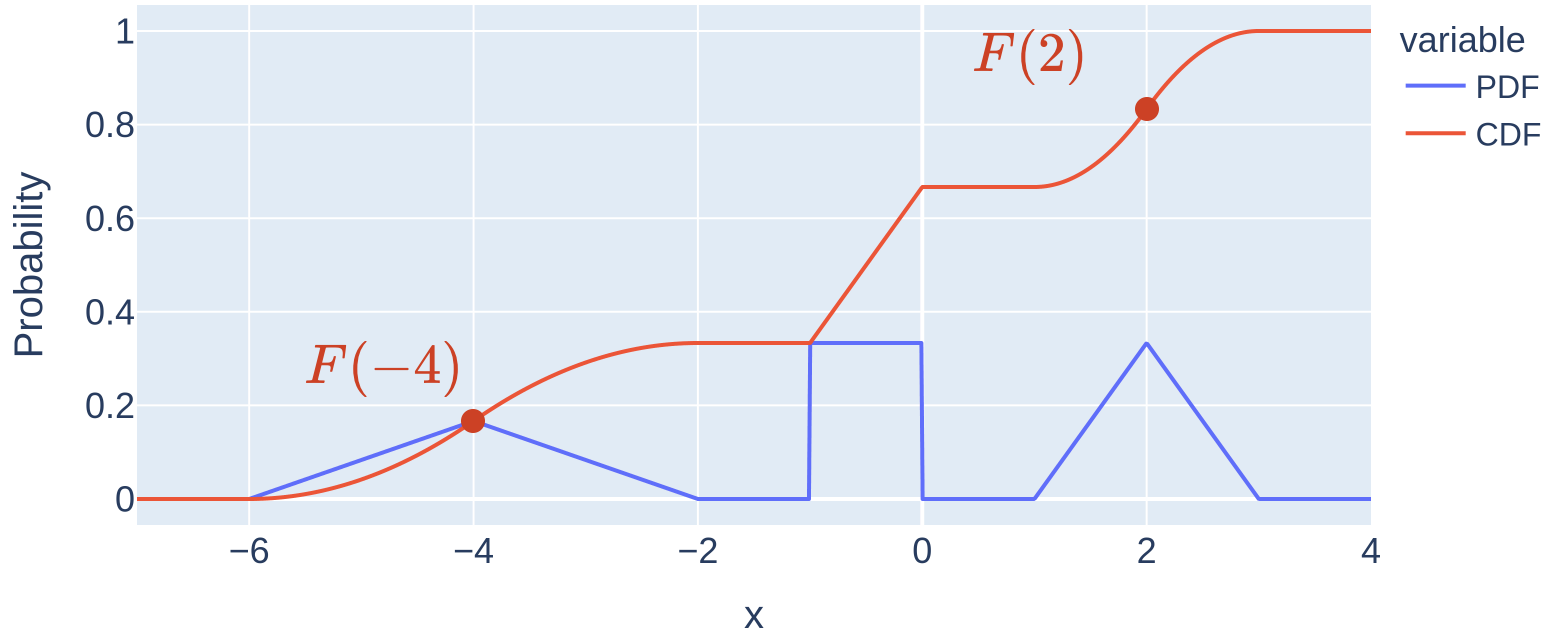
PDF: Probability per unit length

CDF: $P(X \leq x)$

Applying a CDF

Given that CDFs are already the integral of the PDF, we can use the CDF directly to compute the probability of events without integrating.

For example, consider a random variable X with PDF $f(x)$ and CDF $F(x)$ given below:



How would we compute $P(X \in [-4,\ 2])$? $F(2) - F(-4) \approx 0.83 - 0.17 \approx 0.67$

The Exponential Distribution

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Modeling Problem: Waiting Time for an Emission

Suppose we want to model the chance that we'll need to wait more than 3 minutes for an alpha particle. Assume the expected waiting time is 5 minutes.

One natural approach is to define $X \sim \text{Geometric}(0.2)$.

- Conceptually, we are flipping a coin every minute, with $p = 0.2$ chance of emission.
- Then we can compute: $P(X > 3) = (0.8)^3$

But we could also run a more accurate simulation!

- Define $X_{60} \sim \text{Geometric}\left(\frac{0.2}{60}\right)$.
- Now we are flipping 60 coins every minute, with $p = 0.2/60$ chance of emission.
- Now we compute: $P(X_{60} > 60 \cdot 3) = \left(1 - \frac{0.2}{60}\right)^{180}$

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Modeling Problem: Waiting Time for an Emission

Suppose we want to model the chance that we'll need to wait more than 3 minutes for an alpha particle. Assume the expected waiting time is 5 minutes.

One natural approach is to define $X \sim \text{Geometric}(0.2)$.

- Then we can compute: $P(X > 3) = (1 - 0.2)^3$

More accurate approach, define $X_{60} \sim \text{Geometric}\left(\frac{0.2}{60}\right)$

- Now we compute: $P(X_{60} > 180) = \left(1 - \frac{0.2}{60}\right)^{180}$

Can convert X_{60} from number of flips to number of minutes by defining $M_{60} = \frac{X_{60}}{60}$

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Modeling Problem: Waiting Time for an Emission

Suppose we want to model the chance that we'll need to wait more than 3 minutes for an alpha particle. Assume the expected waiting time is 5 minutes.

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- Then we can compute: $P(X > 3) = (1 - 0.2)^3$

More accurate approach, define $X_{60} \sim \text{Geometric}\left(\frac{0.2}{60}\right)$, $M_{60} = \frac{X_{60}}{60}$

- Now we compute: $P(M_{60} > 3) = \left(1 - \frac{0.2}{60}\right)^{180}$

Can convert X_{60} from number of flips to number of minutes by defining $M_{60} = \frac{X_{60}}{60}$

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Modeling Problem: Waiting Time for an Emission

Suppose we want to model the chance that we'll need to wait more than 3 minutes for an alpha particle. Assume the expected waiting time is 5 minutes.

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More accurate approach, define $X_{60} \sim \text{Geometric}\left(\frac{0.2}{60}\right)$, $M_{60} = \frac{X_{60}}{60}$

- Now we compute: $P(M_{60} > 3) = \left(1 - \frac{0.2}{60}\right)^{180}$

Give $P(M_{60000} > 3)$. $= \left(1 - \frac{0.2}{60000}\right)^{60000 \cdot 3}$

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Modeling Problem: Waiting Time for an Emission

Suppose we want to model the chance that we'll need to wait more than 3 minutes for an alpha particle. Assume the expected waiting time is 5 minutes.

One natural approach is to define $X \sim \text{Geometric}(0.2)$.

- Then we can compute: $P(X > 3) = (1 - 0.2)^3$

More accurate approach, define $X_{60} \sim \text{Geometric}\left(\frac{0.2}{60}\right)$, $M_{60} = \frac{X_{60}}{60}$

- Now we compute: $P(M_{60} > 3) = \left(1 - \frac{0.2}{60}\right)^{180}$

$$P(M_{60000} > 3) = \left(1 - \frac{0.2}{60000}\right)^{3 \times 60000}$$

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Result vs. Temporal Resolution

Below, we see the results for different levels of temporal resolution.

- X: Model as one flip per minute.
- M_n: Model as n flips per minute.

Random Variable	Probability of Being Greater than 3	Numerical Value
X	$(1-0.2)^3$	0.512
M_{60}	$\left(1-\frac{0.2}{60}\right)^{3\times 60}$	0.548
M_{60000}	$\left(1-\frac{0.2}{60000}\right)^{3\times 60000}$	0.549
M_n	$\left(1-\frac{0.2}{n}\right)^{3\times n}$	

What happens as $n \to \infty$? We'll use $\lim_{x\to\infty}(1-1/x)^x = e^{-1}$

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Result vs. Temporal Resolution

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M_n	$\left(1-\frac{0.2}{n}\right)^{3\times n}$	

What happens as $n \to \infty$? We'll use $\lim_{x\to\infty}(1-1/x)^x = e^{-1}$

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Closed Form for Infinite Temporal Resolution

We have that $P(M_n > 3) = \left(1 - \frac{0.2}{n}\right)^{3 \times n}$

$$\lim_{x \to \infty} (1 - 1/x)^x = e^{-1}$$

$$= \left(1 - \frac{1}{5} \cdot \frac{1}{n}\right)^{3n}$$

Let $x = 5n$

$$= \left(1 - \frac{1}{x}\right)^{3n} \qquad n = \frac{x}{5}$$

$$= \left(1 - \frac{1}{x}\right)^{\frac{3}{5}x}$$

$$= \left(\left(1 - \frac{1}{x}\right)^{x}\right)^{3/5} = \left(e^{-1}\right)^{3/5} = e^{-3/5}$$

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General Closed Form for Infinite Temporal Resolution

We have that for $\lambda = 0.2$, $P(M_n > 3) = \left(1 - \frac{0.2}{n}\right)^{3 \times n}$

- On the previous slide we showed that $\lim_{n \to \infty} P(M_n > 3) = e^{-3/5}$

$$\frac{P(M_n > a)}{}$$

More generally, if we have arbitrary λ and a, we can find:

$$\lim_{n \to \infty} P(M_n > a) = \left(1 - \frac{\lambda}{n}\right)^{an}$$

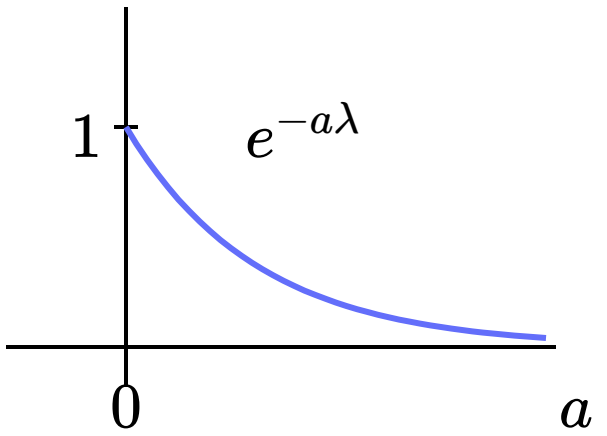
Let $\frac{\lambda}{n} = \frac{1}{x}$

$$= \left(1 - \frac{1}{x}\right)^{an}$$

$n = \lambda x$

$$= \left(1 - \frac{1}{x}\right)^{a\lambda x}$$

$$= \left(\left(1 - \frac{1}{x}\right)^{x}\right)^{a\lambda} = e^{-a\lambda}$$



$$\lim_{x \to \infty} (1 - 1/x)^{x} = e^{-1}$$

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Exponential Distribution

Let Y be the random variable if we crank the temporal resolution of our coin flipping experiment to infinity, i.e. $Y = \lim_{n \to \infty} M_n$. Then we have that:

$$P(Y > a) = \begin{cases} e^{-a\lambda}, & \text{if } a \geq 0 \\ 1, & \text{otherwise} \end{cases}$$

Y is the **exponential distribution**.

- This is a continuous distribution, i.e. $P(Y = a) = 0$ for any specific a.
- The equation above gives the probability of the event $M_\infty > a$.

Let's figure out the CDF and then PDF of this random variable.

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Exponential Distribution

If $Y \sim \text{Exponential}(\lambda)$, then we know:

$$P(Y > a) = \begin{cases} e^{-a\lambda}, & \text{if } a \geq 0 \\ 1, & \text{otherwise} \end{cases}$$

What is the CDF of Y? Reminder, a CDF gives the probability that a random variable is less than or equal to some quantity.

- https://joshh.ug/C



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Exponential Distribution

If $Y \sim \text{Exponential}(\lambda)$, then we know:

$$P(Y > a) = \begin{cases} e^{-a\lambda}, & \text{if } a \geq 0 \\ 1, & \text{otherwise} \end{cases}$$

Since $P(Y \leq a) = 1 - P(Y > a)$, we have that the CDF $F_Y(a)$ is given by:

$$P(Y \leq a) = \begin{cases} 1 - e^{-a\lambda}, & \text{if } a \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

What is the PDF $f_Y(a)$?

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Application of an Exponential Random Variable

Suppose that the number of minutes until the next neutrino detection is given by $Y \sim \text{Exponential}(0.02)$.

What is the chance that the next neutrino detection occurs in the next 100 minutes?

- https://joshh.ug/E

$$F_Y(y) = \begin{cases} 1 - e^{-\lambda y}, & \text{if } y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & \text{if } y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{d\left(1 - e^{-\lambda y}\right)}{dy} = \lambda e^{-\lambda y}$$

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Application of an Exponential Random Variable

Suppose that the number of minutes until the next neutrino detection is given by $Y \sim \text{Exponential}(0.02)$.

What is the chance that the next neutrino detection occurs in the next 100 minutes?

- Easiest to use the CDF! We want $F_Y(100) = P(Y \leq 100)$

$$F_Y(y) = \begin{cases} 1 - e^{-\lambda y}, & \text{if } y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= 1 - e^{-100 \times 0.02}$$

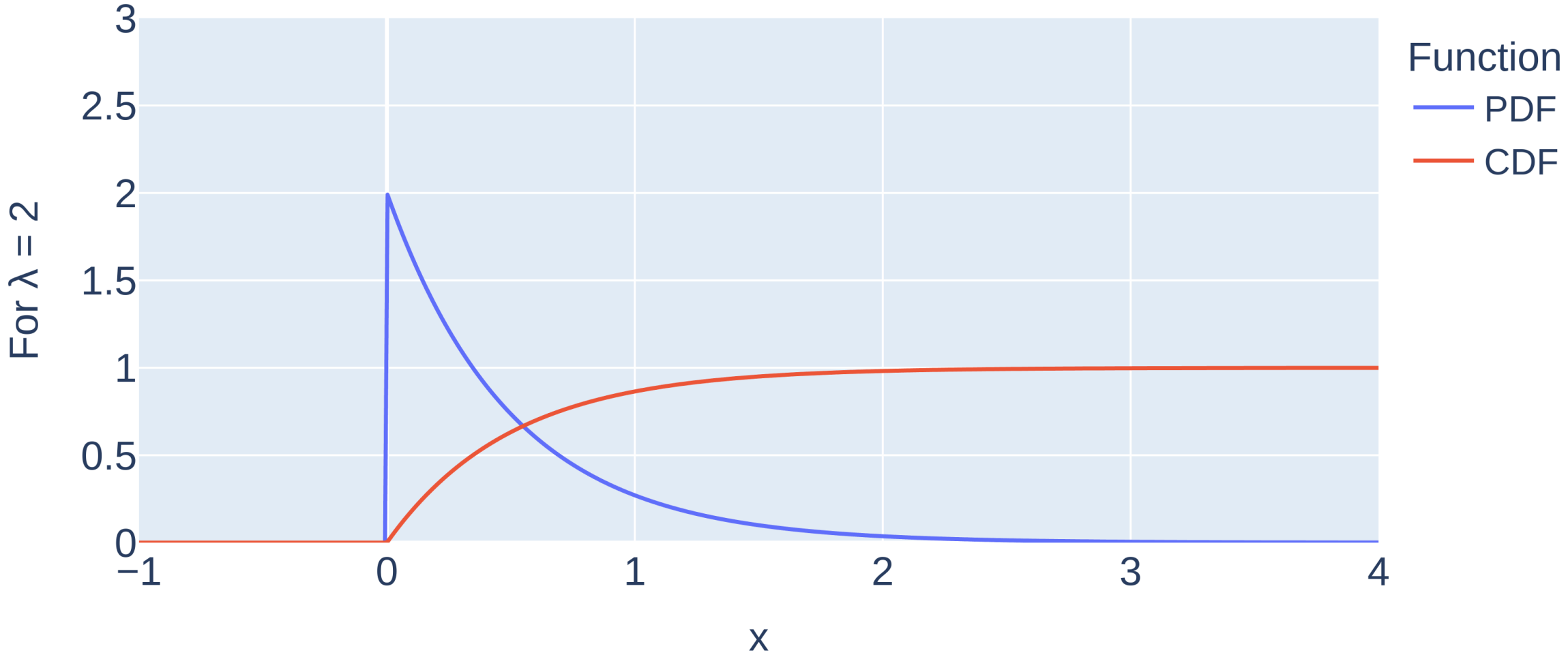
$$= 86.47\%$$

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The PDF and CDF of an Exponential Random Variable (Visually)

Below are the PDF and CDF of $X \sim \text{Exponential}(2)$.

- From the plot, how do we verify the area under the PDF is 1?



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Expectation and Variance

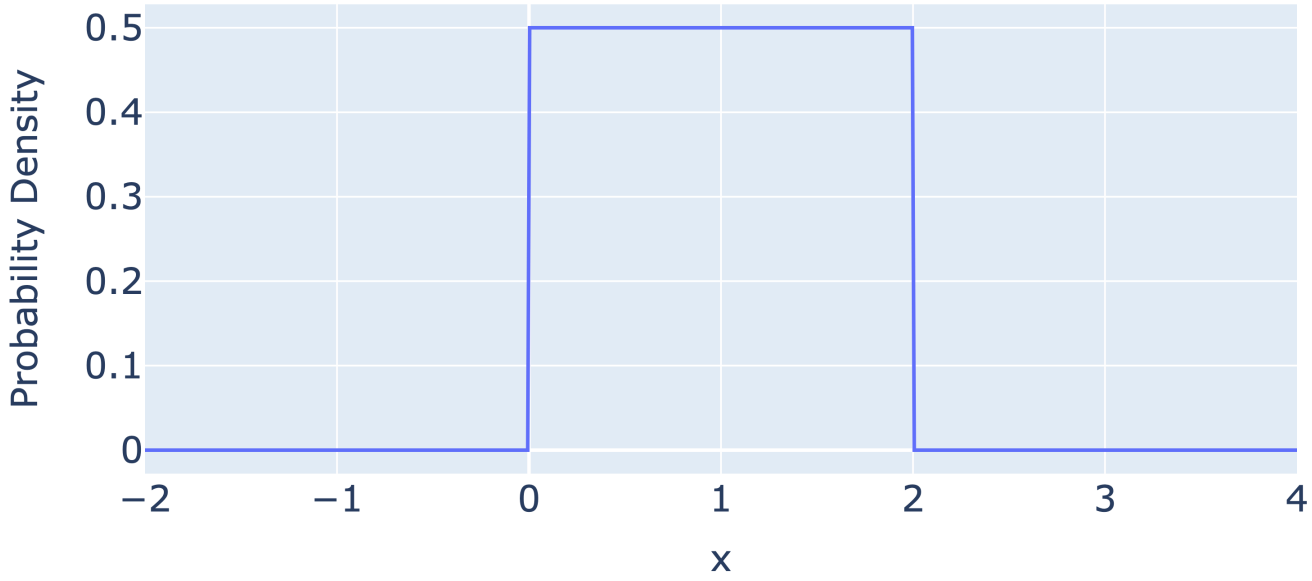
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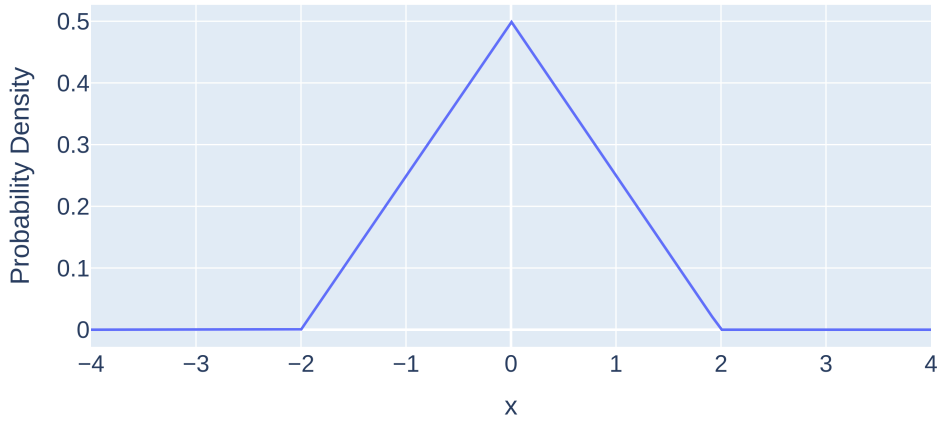
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Expectation of a Continuous Random Variable

What do you think are the expectations of the random variables below?



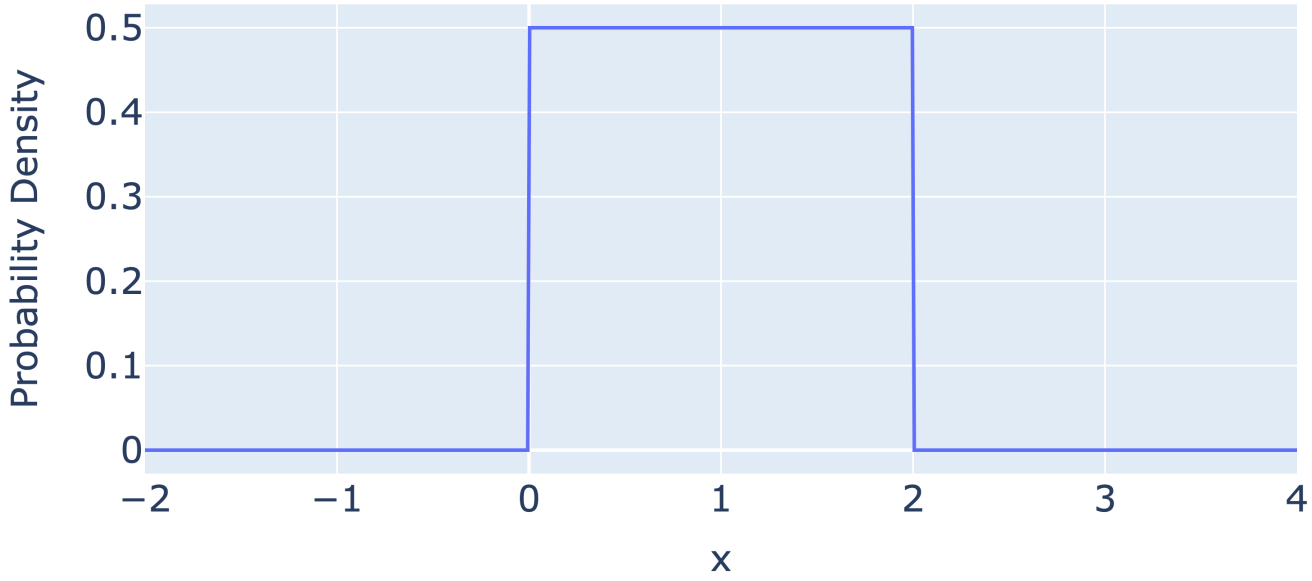


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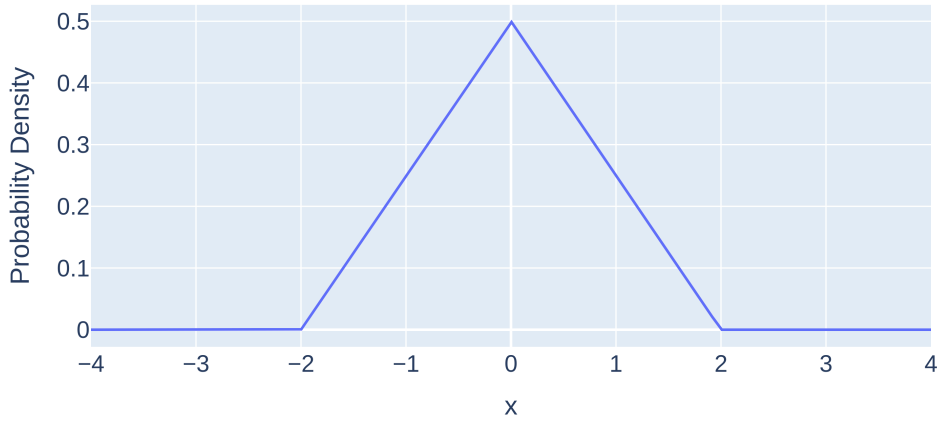
Expectation of a Continuous Random Variable

What do you think are the expectations of the random variables below?

$$\mathbb{E}[X] = 1$$



$$\mathbb{E}[X] = 0$$

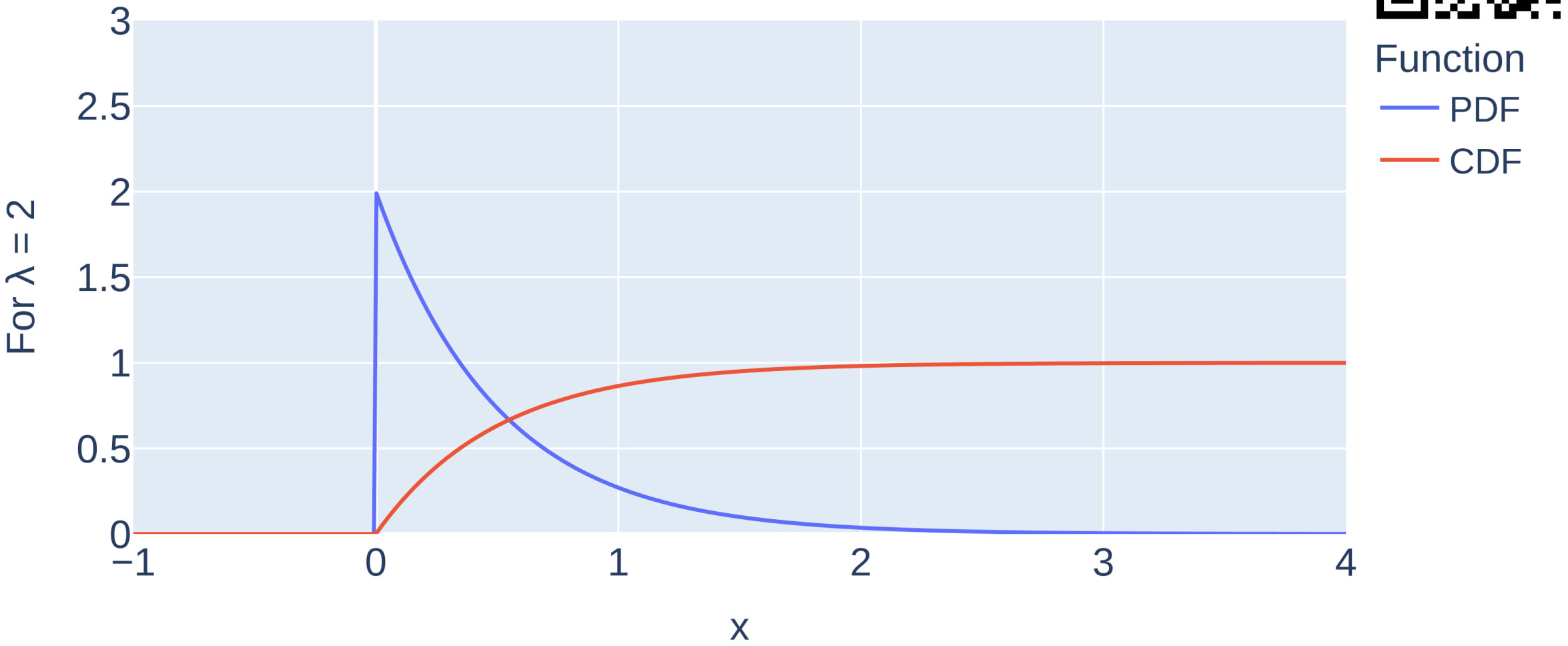


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Expectation of an Exponential Random Variable

What do you think is the expectation of $X \sim \text{Exponential}(2)$?

- https://joshh.ug/F



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Definition of Expectation for a Continuous Random Variable

If X is a random variable with PDF $f_X(x)$, then $\mathbb{E}[X]$ is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

Contrast with our discrete expectation formula:

$$\mathbb{E}[X] = \sum_{x \in \text{range}(X)} x \cdot P(X = x)$$

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Expectation for $X \sim \text{Exponential}(\lambda)$

To compute $\mathbb{E}[X]$, where $X \sim \textbf{Exponential}(\lambda)$, we integrate over the PDF of our exponential random variable.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_0^\infty \underbrace{x}_{u(x)} \cdot \underbrace{\lambda e^{-\lambda x}}_{v'(x)} dx$$

Recall Integration by Parts:

$$\int_a^b u(x) \cdot v'(x) dx = u(x) \cdot v(x)\Big|_a^b - \int_a^b u'(x) \cdot v(x) dx$$

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Expectation for $X \sim \text{Exponential}(\lambda)$

To compute $\mathbb{E}[X]$, where $X \sim \textbf{Exponential}(\lambda)$, we integrate over the PDF of our exponential random variable.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

0 when $x = 0$, 0 when $x = \infty$

$$\mathbb{E}[X] = \int_0^\infty \underbrace{x}_{u(x)} \cdot \underbrace{\lambda e^{-\lambda x}}_{v'(x)} dx = x \cdot (-e^{-\lambda x})\Big|_0^\infty - \int_0^\infty 1 \cdot (-e^{-\lambda x}) dx$$

$$= 0 \qquad - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$

When $x = \infty$ it's 0, when $x = 0$ it's $-\frac{1}{\lambda}$

$0 + \frac{1}{\lambda}$

Recall Integration by Parts:

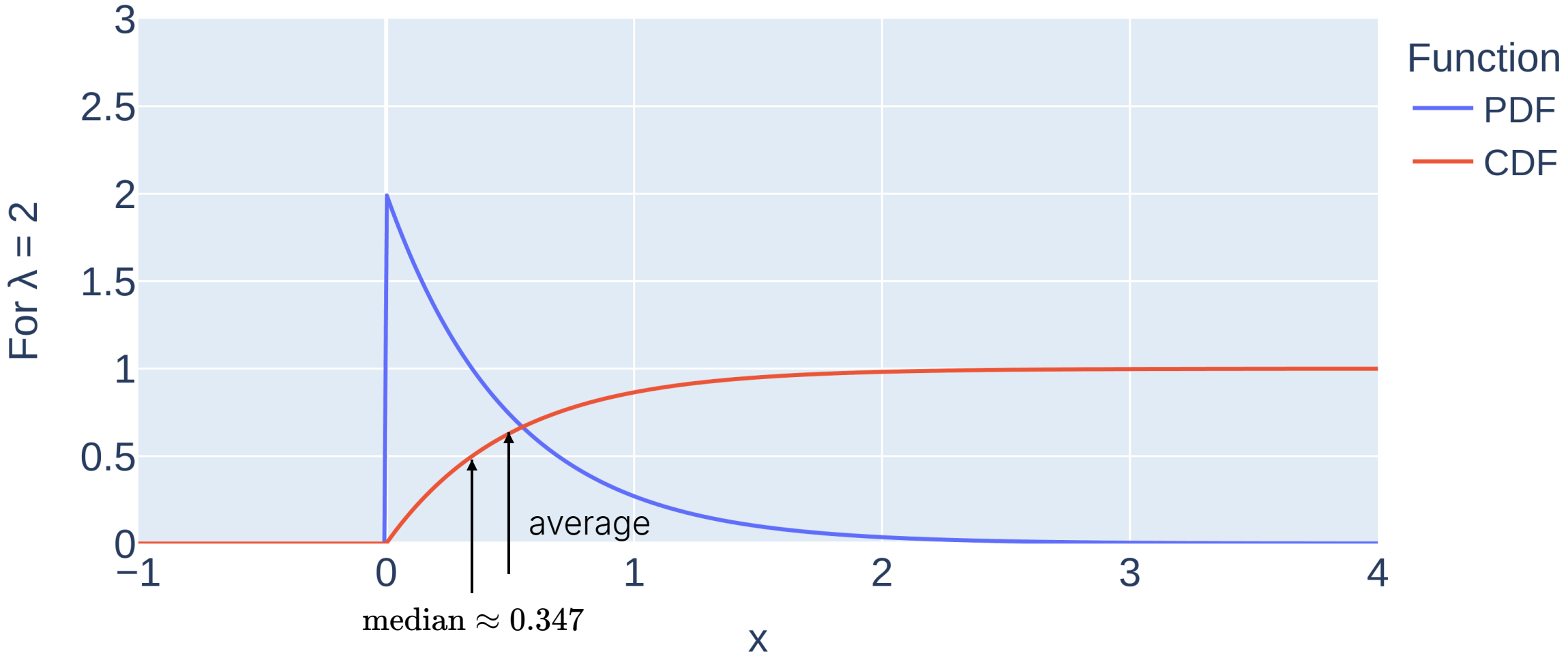
$$\int_a^b u(x) \cdot v'(x) dx = u(x) \cdot v(x)\Big|_a^b - \int_a^b u'(x) \cdot v(x) dx$$

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Expectation of an Exponential Random Variable

What do you think is the expectation of $X \sim \text{Exponential}(2)$?

- Answer: $1/\lambda = 1/2$



Definition of Variance for a Continuous Random Variable

If X is a random variable with PDF $f_X(x)$, then $\mathbf{Var}(X)$ is:

$$\mathbf{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \underbrace{\int_{-\infty}^{\infty} x^2 \cdot f_X(x) dx}_{\text{LOTUS}} - \underbrace{\left(\int_{-\infty}^{\infty} x \cdot f_X(x) dx \right)^2}_{(\mathbb{E}[X])^2}$$

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Variance for $X \sim \text{Exponential}(\lambda)$

To compute the variance, we'll need $\mathbb{E}[X^2]$. Luckily this isn't so bad.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_0^\infty \lambda x e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \int_0^\infty \underbrace{x^2}_{u(x)} \cdot \underbrace{\lambda e^{-\lambda x}}_{v'(x)} dx = \overbrace{\underbrace{x^2 \cdot (-e^{-\lambda x})\big|_0^\infty}_{0}}^{0 \to \infty \quad -1 \to 0} - \int_0^\infty 2x \cdot (-e^{-\lambda x}) dx$$

$$= 0 \qquad + \frac{2}{\lambda} \int_0^\infty \underbrace{\lambda x e^{-\lambda x} dx}_{\mathbb{E}[X] = \frac{1}{\lambda}} = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

Recall Integration by Parts:

$$\int_a^b u(x) \cdot v'(x) dx = u(x) \cdot v(x)\big|_a^b - \int_a^b u'(x) \cdot v(x) dx$$

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$$\mathbb{E}[X] = \int_0^\infty \lambda x e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

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Geometric vs. Exponential

Compare $C \sim \text{Exponential}(\lambda)$ and $D \sim \text{Geometric}(\lambda)$

$$\text{PDF: } f(c) = \lambda e^{-\lambda c}$$

$$\text{PMF: } \overbrace{(1-\lambda)^{d-1}}^{d-1 \text{ tails}}\lambda \leftarrow \text{one head}$$

$$\mathbb{E}[C] = \frac{1}{\lambda}$$

$$\mathbb{E}[D] = \frac{1}{\lambda}$$

$$\text{Var}(C) = \frac{1}{\lambda^2}$$

$$\text{Var}(D) = \frac{1-\lambda}{\lambda^2}$$

$$\lambda \in (0, \infty)$$

instantaneous chance of happening now

$\lambda = 1/\text{expected time}$

e.g. $\text{Exponential}\left(0.1 \ \frac{1}{\text{minutes}}\right)$

$$\lambda \in (0, 1]$$

probability of the coin being heads

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Expectation and Variance of a Uniform Random Variable

In the notes, they show that if X is a uniform random variable over $[0, \ell]$:

$$\mathbb{E}[X] = \frac{\ell}{2} \qquad \mathrm{Var}(X) = \frac{\ell^2}{12}$$

See notes (or just show yourself) for a proof. Contrast with D uniform over $\{1, 2, \ldots, n\}$, which was:

$$\mathbb{E}[D] = \frac{n+1}{2} \qquad \mathrm{Var}(D) = \frac{n^2 - 1}{12}$$

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Joint Distributions and Independence

Lecture 24, CS70 Fall 2026

Continuous Probability Distributions
The Exponential Distribution
Expectation and Variance
Joint Distributions and Independence
Summary and Reflections

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Joint Distributions and Independence

Our usual ideas from discrete probability around joint distributions, marginal distributions, and independence carry over very closely.

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Review: Joint Distributions (Discrete)

To refresh our memory on joint distributions, let's first consider the joint PMF of two discrete uniform random variables X and Y representing the roll of two independent four-sided dice.

What is the joint distribution of X and Y?

$\{((1,1),\ 1/16), ((1,2),\ 1/16), ((1,3),\ 1/16), \ldots\}$

x	y	$P(X = x, Y = y)$
1	1	1/16
1	2	1/16
1	3	1/16
1	4	1/16
2	1	1/16
2	2	1/16
2	3	1/16
2	4	1/16
3	1	1/16
3	2	1/16
3	3	1/16
3	4	1/16
4	1	1/16
4	2	1/16
4	3	1/16
4	4	1/16

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Joint PMF

Suppose we have random variables X and Y which are discrete. Then their joint PMF is a non-negative function $P(X = x, Y = y)$ such that:

$$\sum_{y \in \text{range}(Y)} \sum_{x \in \text{range}(X)} P(X = x, Y = y) = 1$$

And for all $a < b$ and $c < d$, we have:

$$P(a \leq X \leq b,\ c \leq Y \leq d) = \sum_{c \leq y \leq d} \sum_{a \leq x \leq b} P(X = x, Y = y)$$

$P(X = x, Y = y) =$ probability of landing exactly on the point (x, y).

- This is an actual probability, not a density.

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Joint Density

Suppose we have random variables X and Y which are continuous. Then their joint PDF is a non-negative function $f_{X,Y}(x,y)$ such that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

And for all $a < b$ and $c < d$, we have:

$$P(a \le X \le b,\ c \le Y \le d) = \int_{c}^{d}\int_{a}^{b} f_{X,Y}(x,y)\,dx\,dy$$

$f_{X,Y}(x,y)\,dx\,dy \approx$ probability of landing in a $dx \times dy$ rectangle around (x,y).

- Actual probability is zero since the rectangle is infinitesimal.

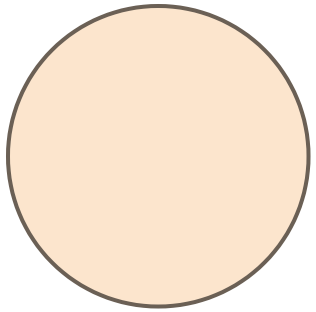
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Joint Density Example

Suppose we have random variables X and Y which are continuous. Then their joint PDF is a non-negative function $f_{X,Y}(x,y)$ such that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Suppose we throw a dart at the origin. Suppose we land uniformly within 2 feet of the origin. What is c? https://joshh.ug/G



2 foot radius

$$f_{X,Y}(x,y) = \begin{cases} c, & \text{if } x^2+y^2 \le 4 \\ 0, & \text{otherwise} \end{cases}$$

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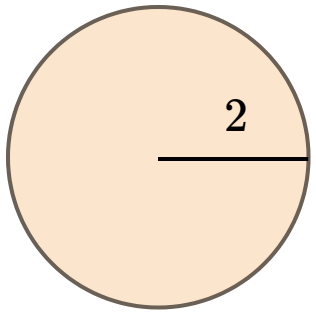
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$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Suppose we throw a dart at the origin. Suppose we land uniformly within 2 feet of the origin.

$$\text{area} = \pi r^2 = \pi \cdot 2^2 = 4\pi$$



2 foot radius

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{4\pi}, & \text{if } x^2+y^2 \le 4 \\ 0, & \text{otherwise} \end{cases}$$

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Marginal Probabilities

Suppose we have random variables X and Y which are continuous. Then their joint PDF is a non-negative function $f_{X,Y}(x, y)$ such that:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$$

Similar to marginal probabilities for discrete random variables, we can compute $f_X(x)$ and $f_Y(y)$. How?

Discrete: $P(X = x) = \sum_{y \in \text{range}(Y)} P(X = x, Y = y)$

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Marginal Probabilities

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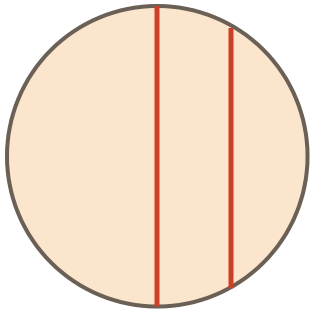
Discrete: $P(X = x) = \sum_{y \in \text{range}(Y)} P(X = x, Y = y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$

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Marginal Density Example

X and Y are jointly distributed as:



2 foot radius

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4\pi}, & \text{if } x^2 + y^2 \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$x = 0$: $y \in [-2, 2]$
$x = 1$: $y \in [-\sqrt{3}, \sqrt{3}]$
$x = 2$: $y \in [0, 0]$

What is the distribution of X, i.e. what is the p.d.f. $f_X(x)$?

First idea: $f_X(x) = \int_{\min y}^{\max y} \frac{1}{4\pi}\, dy$

$y^2 \le 4 - x^2 \Rightarrow y \in \left[-\sqrt{4-x^2},\ \sqrt{4-x^2}\right]$

$$= \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{4\pi}\, dy$$

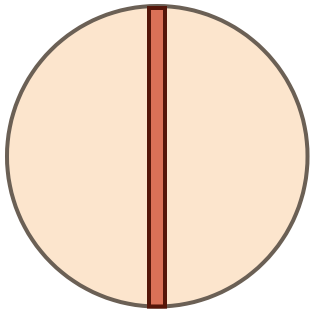
$$= \frac{y}{4\pi}\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} = \frac{\sqrt{4-x^2} + \sqrt{4-x^2}}{4\pi} = \frac{\sqrt{4-x^2}}{2\pi}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

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Marginal Density Example

X and Y are jointly distributed as:



$\approx \frac{\sqrt{4}}{2\pi}\, dx$

2 foot radius

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{4\pi}, & \text{if } x^2 + y^2 \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

What is the distribution of X, i.e. what is the p.d.f. $f_X(x)$?

$$f_X(x) = \begin{cases} \dfrac{\sqrt{4 - x^2}}{2\pi}, & \text{if } x \in [-2,\ 2] \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(0) = \frac{\sqrt{4}}{2\pi}$$

$$f_X(1) = \frac{\sqrt{3}}{2\pi}$$

$$f_X(2) = 0$$

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Independent Random Variables

We can define X and Y to be independent continuous random variables if for all $a < b$ and $c < d$, the following holds true:

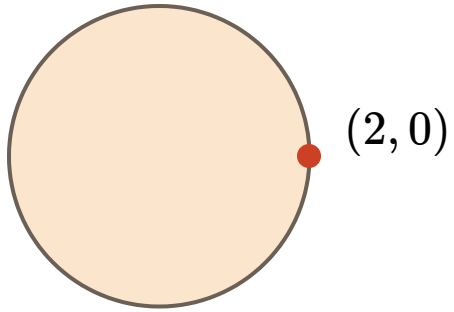
$$P(a \leq X \leq b,\ c \leq Y \leq d) = P(a \leq X \leq b) \cdot P(c \leq Y \leq d)$$

Equivalently (proof omitted): X and Y are independent if and only if $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ for all $x,\ y$.

For the dart example earlier, were X and Y independent?

- Why or why not?

No! $f_{X,Y}(2, 0) = \frac{1}{4\pi}$, but $f_X(2) \cdot f_Y(0) = 0 \cdot \frac{1}{\pi} = 0$.



2 foot radius

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Independent Random Variables

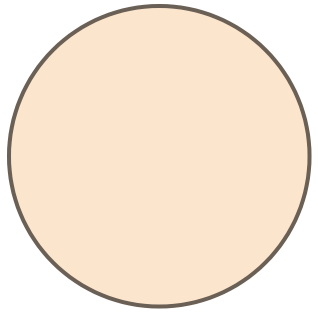
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$$P(a \leq X \leq b,\ c \leq Y \leq d) = P(a \leq X \leq b) \cdot P(c \leq Y \leq d)$$

Equivalently (proof omitted): X and Y are independent if and only if $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ for all $x,\ y$.

For the dart example earlier, were X and Y independent?

- Not independent! For example, if $X = 2$, then Y must be 0.



2 foot radius

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Summary and Reflections

Lecture 24, CS70 Fall 2026

Continuous Probability Distributions
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Discrete vs. Continuous Probability Distributions

Before today, our random variables had a discrete range, i.e. either finite or countably infinite. Example: $\text{range}(\text{Geometric}) = \{1, 2, \ldots\}$

- We considered the **space of sample points first**, e.g. $TTTTTH$, and later defined events in terms of sums of probabilities of sample points.
- Example, if $X \sim \text{Binomial}(3, 0.5)$, then:
  - $P(X = 1) = P(\omega = HTT) + P(\omega = THT) + P(\omega = TTH)$
- The probability of an event defined by random variable equality was given by the "probability mass function". We did not use this term before today.
  - Example: PMF of X above is $P(X = k) = \binom{3}{k} 0.5^k 0.5^{3-k}$ for $k \in \{0, 1, 2, 3\}$

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Discrete vs. Continuous Probability Distributions

Before today, our random variables had a discrete range, i.e. either finite or countably infinite. Example: $\text{range}(\text{Geometric}) = \{1, 2, \ldots\}$

- We considered the **space of sample points first**, e.g. $TTTTTH$, and later defined events in terms of sums of probabilities of sample points.

By contrast, for continuous probability distributions, we think about **events first**.

- CDF gives $P(X \leq a)$, which is the probability of the event that X is less than or equal to a.
- Probability of any specific outcome is zero!
- More generally, we only ever care about the probability of intervals (or unions of intervals).
- PDF is the continuous analog of the PMF, but PDF outputs are not probabilities.

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