Announcements!

- Lecture is posted under “Media Gallery” in bCourses
- **HW 1** and **Vitamin 1** have been released, due **Today** (grace period Friday)

question 4
What is induction?

Goal in induction is to prove some statement for all natural numbers $\forall n \in \mathbb{N}, P(n)$

Principle of Induction

- **Base Case**: Prove $P(0)$
- **Inductive Hypothesis**: Assume $P(n)$
- **Inductive Step**: Prove $P(n) \Rightarrow P(n+1)$
Visual Analogy

Prove all the dominos fall down

- \( P(0) = \text{“First domino falls”} \)  
- \( P(k) \Rightarrow P(k+1) \) \( k \)th domino falls implies that \( k+1 \)st domino falls

Even if you had infinite dominos lined up, this method would prove all of them will fall down (More on this Week 4).
Simple Induction (Example 1)

Theorem: For all natural numbers $n$, $0 + 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

Proof:

**Base Case:** $n=0$ \quad $0 = \frac{0(0+1)}{2} = 0$ \quad $✓$

**Ind. Hyp.:** Assume for some $n=k \geq 0$ it is true that $0+1+\ldots+k = \frac{k(k+1)}{2}$

**Ind. Step:** Prove that for $n=k+1$ the claim holds

$1 + 2 + \ldots + (k+1) = \frac{(k+1)(k+2)}{2}$

$\left(1 + 2 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2+k+2k+2}{2} = \frac{(k+1)(k+2)}{2}\right)$

The second equality holds from the inductive hypothesis. Thus, the theorem holds by induction.
Simple Induction (Example 2)

Theorem: For all $n \in \mathbb{N}$, $3 \mid (n^3 - n)$

Proof:

We induct on the variable $n$.

**Base Case:** $n = 0$  $3 \mid 0^3 - 0$. This is trivially true.

**Inductive Hypothesis (Ind. Hyp.):** For $n = k$, assume $3 \mid k^3 - k$ i.e. $\exists q \in \mathbb{Z}$ s.t. $k^3 - k = 3q$.

**Inductive Step:** We wish to show that for $n = k + 1$, $3 \mid (k+1)^3 - (k+1)$.

$$(k+1)^3 - (k+1) = 3p \quad p \in \mathbb{N}$$

$$(k^3 + 3k^2 + 3k + 1) - (k+1) = 3p$$

$$(k^3 - k + 3k^2 + 3k + 1) = 3p$$

$$3q + 3k^2 + 3k$$

From the In. Hyp.,

$$3\left(q + \frac{k^2 + k}{3}\right) = 3p$$

by Def. It follows that $(k+1)^3 - (k+1)$ is divisible by 3. $\Box$
Simple Induction (Example 3)

Theorem: Any map formed by dividing the plain into regions by drawing straight lines can be properly colored with two colors.

Proof:

We will induct on the number of lines, let \( n \) be \# of lines.

**Base Case:** \( n = 0 \) color the whole plain one color.

**Ind Hyp:** For \( n=k \) lines assume it is two colorable.

**Ind Step:** Consider an arbitrary map with \( k+1 \) lines.

1. Remove one line from the map. By Ind. Hyp. this new map with \( k \) lines is two colorable.
2. Add back the line that was removed and flip all the colors on one side of the line.

By construction all the regions adjacent to the line that was added have different colors. Thus, the region that was not flipped is correctly colored by Hyp. That was flipped, is also two colored by hypothesis since we just changed the labels.
Improving Induction Hypothesis (Example 1)

Theorem: The sum of the first $n$ odd numbers is a perfect square

Improved: The sum of the first $n$ odd numbers is $n^2$

Proof:

**Base Case** $n=1$  \hspace{1cm} 1 = 1^2  \\

**Ind Hyp:** Assume \( \sum_{k=1}^{n} (2k-1) = k^2 \) \hspace{1cm} \text{first } k \text{ odds}

**Ind Step:** Wish to show \( \sum_{k=1}^{n+1} (2k-1) = (n+1)^2 \)

\[
\begin{align*}
    \sum_{k=1}^{n+1} (2k-1) &= \sum_{k=1}^{n} (2k-1) + (2(n+1)-1) \\
    &= n^2 + 2n + 1 \\
    &= (n+1)^2
\end{align*}
\]

by hyp., \hspace{1cm} \square
Improving Induction Hypothesis (Example 2)

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$

Improved:

Proof:
What is Strong Induction?

Principle of Strong Induction

- **Base Case:** Prove $P(0)$
- **Inductive Hypothesis:** Assume $P(0)$ and $P(1)$ and ... and $P(n)$
- **Inductive Step:** Prove $P(0)$ and ... and $P(n) \Rightarrow P(n+1)$

$$P(0) \land P(1) \land \ldots \land P(n) \Rightarrow P(n+1)$$

*Strong induction is implied by weak induction*
Strong Induction (Example 1)

Theorem: Every natural number greater than 1 can be written as a product of one or more primes.

Proof:

**Base Case:** \( n = 2 \). 2 is prime so its prime factorization is just 2.

**Ind. Hyp**: Assume claim holds for all \( 1 \leq n \leq k \).

**Ind. Step**: Let \( n = k+1 \).

**Case 1**: \( k+1 \) is prime. we are done.

**Case 2**: \( k+1 \) is composite. Therefore \( \exists a, b \in \mathbb{N}, k+1 = a \cdot b \) since \( k+1 > 1 \Rightarrow 1 \leq a, b \leq k \). Then by the ind. hyp. \( a \) and \( b \) can be written as a product of primes. Thus \( k+1 \) can be written as a product of \( a \)'s and \( b \)'s primes.

\( \square \)
Strong Induction with Multiple Base Cases (Example 2)

Theorem: For every natural number \( n \geq 12 \), it holds that \( n = 4x + 5y \) for some \( x, y \in \mathbb{N} \).

Proof:

**Base Cases**
- \( n = 12 \): \( 12 = 4(3) + 5(0) \), \( x = 3, y = 0 \)
- \( n = 13 \): \( 13 = 4(2) + 5(1) \)
- \( n = 14 \): \( 14 = 4(1) + 5(2) \)
- \( n = 15 \): \( 15 = 4(0) + 5(3) \)

**Ind Hyp:** Assume claim holds for all \( 12 \leq n \leq k \).

**Ind Step:** \( n = k + 1 \geq 16 \). Then, \( (k+1) - 4 \geq 12 \).

By the Ind. hyp. \( (k+1) - 4 = 4x' + 5y' \) for some \( x', y' \in \mathbb{N} \).

\[ k + 1 = 4x' + 5y' + 4 = 4(x' + 1) + 5y'. \]

So, then we can set \( x = x' + 1 \) and \( y = y' \).

\( k + 1 = 4x + 5y \)
Why ever use weak induction?

Weak Induction $\Rightarrow$ Strong Induction

If you wanted to you could always use strong induction

It is nicer to only use weak induction if strong induction is not needed.

- It's easier for the reader
- Easier to catch mistakes
Well-Ordering Principle

The Well-Ordering Principle states that for any non-empty subset of the natural numbers there will be a least element.

Theorem: Every natural number greater than 1 can be written as a product of one or more primes

Proof using WOP:

Let \( S \) be the set of natural numbers that cannot be written as a product of primes. Assume for contradiction that \( S \) is not empty.

By WOP, \( S \) has a least element \( n \).

Clearly, \( n \) is not prime. So, we can write \( n = a \cdot b \) as well.

It follows that \( a \) or \( b \) doesn't have a prime factorization.

Without loss of generality (WLOG) say \( a \) can't be written as a product of primes. Notice, since \( n > 1 < a < n \). This is a contradiction because then \( a \in S \), but we said \( n \) is the least element!

Thus, \( S \) is empty and theorem holds. \( \square \)
Summary

- **Simple Induction**
  - P(0) and show P(n) \(\Rightarrow\) P(n+1)

- **Multiple Base Cases**
  - You may need multiple base cases to prove a statement

- **Improve the Inductive Hypothesis**
  - Sometimes proving a “stronger” statement is easier

- **Strong Induction**
  - P(0) and show P(0) and ... and P(n) \(\Rightarrow\) P(n+1)

- **Well Ordering Principle**
  - For any subset of the naturals there is a least element