Announcements!

- Read the Weekly Post
- Tarang’s OH 4-6p in Woz Lounge (Zoom also—same link as lecture)
  - First 30 minutes for conceptual question
  - Last 90 minutes for reading Note 5 together and question about the note
  - Will not prioritize HW questions. Use regular OH for that.
- **HW 2** and **Vitamin 2** have been released, due **Thu** (grace period Fri)
- We are adding a bit more OH support, but also work on the HW early
- Throughout this lecture **definitions** will be underlined

UC Berkeley EECS 70 - Tarang Srivastava
Undirected Simple Graph Definitions

An undirected simple graph $G = (V, E)$ is defined by
1. A set $V$ of vertices. Sometimes we may call it a node.
2. A set $E$ of edges
Where edges in $E$ are of the form $\{u, v\}$ for $u, v \in V$ and $u \neq v$.
A graph being simple here means no parallel edges.
A graph being undirected means there's no direction to the edges.
Examples:
Directed Graph Definitions

Edges in a **directed graph** are defined as \((u, v)\). That is, the order of the vertices matters. Therefore, \((u, v) \neq (v, u)\).

Examples:
Edge and Degree Definitions

Given an edge $e = \{u, v\}$ we say

- $e$ is **incident** to $u$ and $v$
- $u$ and $v$ are **neighbors**
- $u$ and $v$ are **adjacent**
- The **degree** of a vertex $v$ is the number of incident edges
  - $\text{deg}(v) = |\{v \in V \mid \{u, v\} \in E\}|$

Examples:
Summary Questions I

How many nodes in this graph? 11

How many edges? 16

Which vertex has the max degree? 5, 8, 2, 10, 1

Which vertex has the min degree? 6, 1, 3, 9, 7, 1

Which vertices is this edge incident on? 2, 5

What is the sum of the degrees? 32
Handshake Lemma

Lemma: The sum of the degree of all the vertices is equal to $2|E|$

Proof: Proceed by induction on $|E| = m$

**Base Case:** $m = 0$. A graph has no edges if all the vertices are isolated (i.e. no neighbors) thus each vertex is degree 0

**Ind Hyp:** Assume claim holds for $m = k$ edges, ... sum of degrees is $2K$

**Ind Step:** Consider an arbitrary graph $G$ with $k+1$ edges. Remove any edge from $G$. The new graph has $k$ edges, and by the inductive hypothesis sum of degrees is $2K$. Then adding back the edge we add 1 degree to each incident vertex. Thus sum of degrees is now $2k + 2 = 2(k+1)$ as desired.
Path, Cycles, Walks and Tours

**Deals with Vertices** (though may imply things about edges):

**Path:** A sequence of vertices in $G$, generally with no repeated vertices.

**Cycle:** A path in $G$ where the only repeated vertex is the first one and last one.

**Deals with Edges** (though may imply things about vertices):

**Walk:** Is a sequence of edges with possible repeated vertex or edges.

**Tour:** A walk that starts and ends at the same vertex.

**Eulerian walk:** A walk where each edge is visited exactly once.

**Eulerian tour:** An Eulerian walk that starts and ends at the same vertex.
Summary Questions II

Give an example of length 3 cycle? $\{3, 11, 4\}$

Give an example of a path from 2 to 8? $2, 5, 8$

Give the longest simple path? $4, 3, 1, 5, 8, 7, 10, 1, 2, 8$

How many connected components are there?

Give an example of length 4 tour? $7 \to 10 \to 5 \to 8$
Connectivity

A graph $G$ is said to be **connected** if there exists a path between any two vertices.

Examples:

Any graph always consists of a collections of **connected components**. A connected component is a set of vertices in the graph that are connected.
Eulerian Tours

**Eulerian walk:** A walk where each edge is visited exactly once.

**Eulerian tour:** An Eulerian walk that starts and ends at the same vertex.

Theorem: A undirected graph $G$ has an Eulerian tour iff $G$ is even degree, and connected.

Proof: in the notes
Summary Questions III

Is there an Eulerian Tour and if so provide a tour?

Why? Every vertex has even degree & connected

How many connected components now? 1

Connected components now? 1

What about now? 4
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$

Base Case: There is only one graph with a single vertex and it has degree 0. Thus, vacuously true.

Inductive Hypothesis: Assume the claim is true for $n \geq 1$

Inductive Step: We prove the claim is also true for $n + 1$. Consider an undirected graph with $n$ vertices and each has degree greater than 1. By the inductive hypothesis, this graph is connected.

Now add one more vertex $x$ to obtain a graph with $(n + 1)$ vertices.

Since, the previous graph was connected, and $x$ is connected to some node $y$ then there’s a path between $x$ and any other vertex through $y$, since by definition there’s a path from $y$ to any other vertex. Thus, the graph is connected.
Minimum Edges for Connectivity

Theorem: Any connected graph with \( n \) vertices must have at least \( n-1 \) edges

**Induction on vertices** \( n = |V| \)

**Base Case:** \( n=1 \) 0 edges 0 = (1) - 1

**Ind Step:** Assume claim holds for \( 1 \leq n \leq k \)

**Ind Step:** Consider a connected graph \( G \) with \( n = k+1 \) vertices

Remove an arbitrary vertex \( u \). Removing \( u \) suppose \( G \) now is \( k \) connected components. By strong induction each connected component \( G_1, G_2, \ldots, G_k \) has \( k_i - 1 \) edges, \( k_2 - 1 \) edges \ldots \( k_k - 1 \) edges. Adding back vertex \( u \), we have \( 1 \leq |E| \) to add back \( S \) edges

\[ |E| \geq k - S \]

\[ (k+1) - 1 \]

\[ 1 \leq |E| \geq k \] as done
Complete Graphs

A graph $G$ is **complete** if it contains the maximum number of edges possible.

Examples:
Trees

The following definitions are all equivalent to show that a graph $G$ is a **tree**.

1. $G$ is connected and contains no cycles
2. $G$ is connected and has $n-1$ edges (where $n = |V|$)
3. $G$ is connected, and the remove of any single edge disconnects $G$
4. $G$ has no cycles, and the addition of any single edge creates a cycle
Tree Definitions are Equivalent

Theorem: For a connected graph $G$ it contains no cycles iff it has $n-1$ edges.

Proof:
Tree Definitions are Equivalent (cont.)

Theorem: For a connected graph $G$ it contains no cycles iff it has $n-1$ edges.
Bipartite Graphs

A graph $G$ is **bipartite** if the vertices can be split in two groups (L or R) and edges only go between groups.

Examples: