Countability review

A, B countable

$A \cup B$ countable

$A \times B$ countable

$\bigcup_{i=1}^{\infty} A_i$ countable

A countable, B infinite, $B \subset A \Rightarrow B$ countable

A^B (infinite sequences) : maybe not countable

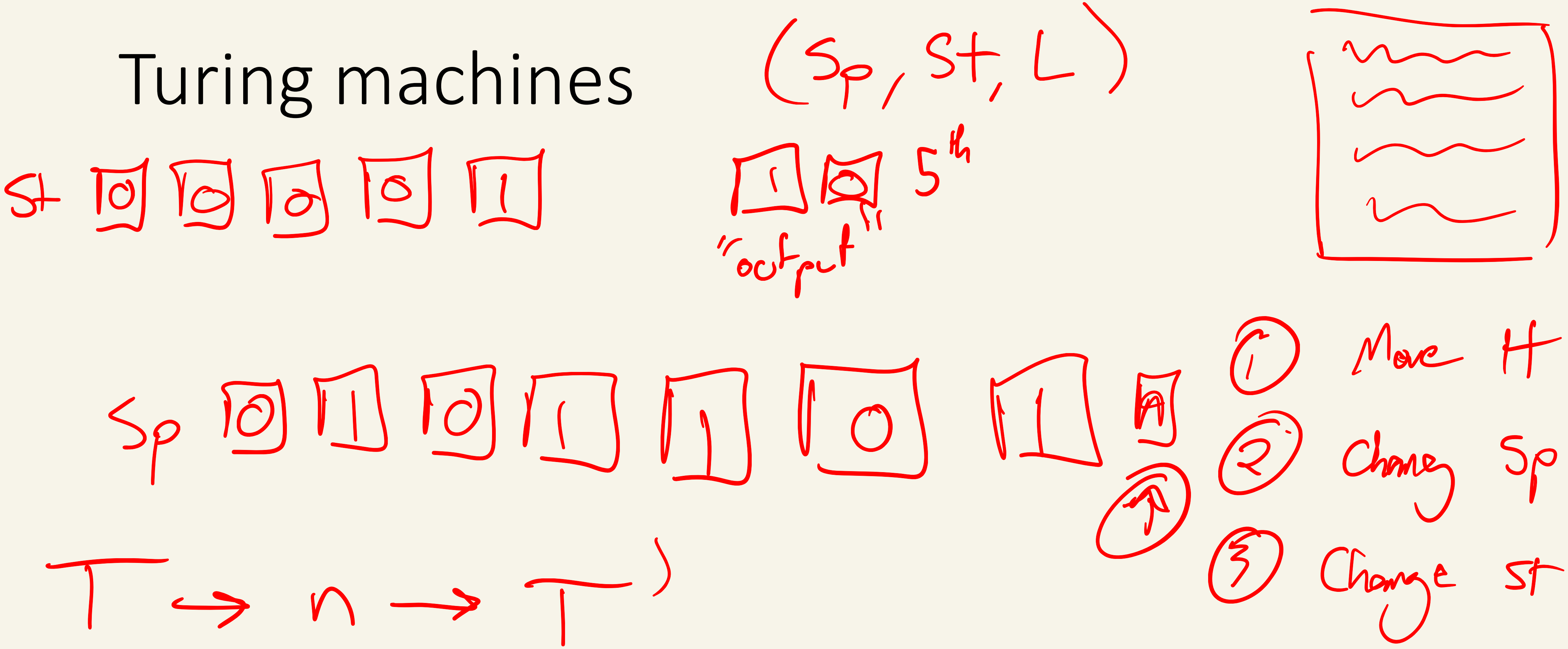
$\mathbb{Z}^{\times k}$ $(a, b, c, \ldots)$ / $\mathbb{Z}^{\mathbb{N}}$ $\{0,1\}^{\mathbb{N}}$, $\mathbb{N}^{\{0,1\}}$ $(4, 7)$

Computability

The theory of computation

Michael Psenka

Turing machines



Church-Turing Thesis: "any $f : \mathbb{N} \rightarrow \mathbb{N}$ computable by an effective method is computable by a Turing machine"

Computable sets

$S \subset \mathbb{N}$ is <u>computable</u> if $\exists$ Turing machine (S_p, St, L) s.t.

$$a \in S \iff f_T(a) = 1$$

$f_T(a) :=$ output of (S_p, St, L) w/ input a

e.g. $f_T(a) = 2^a$

$S := \{$ "computable sets" $\}$

$A \in S$ $\exists$ (s_1, St, L) st f_T computes A

$F: S \to (s_p, St, L)$, "1st Turing machine that computes A"

How big is the collection of computable sets?

Turing Machines countable?

S subset of countable set $\Longrightarrow$ countable. ☒

Quines, recursion theorem

Fun a

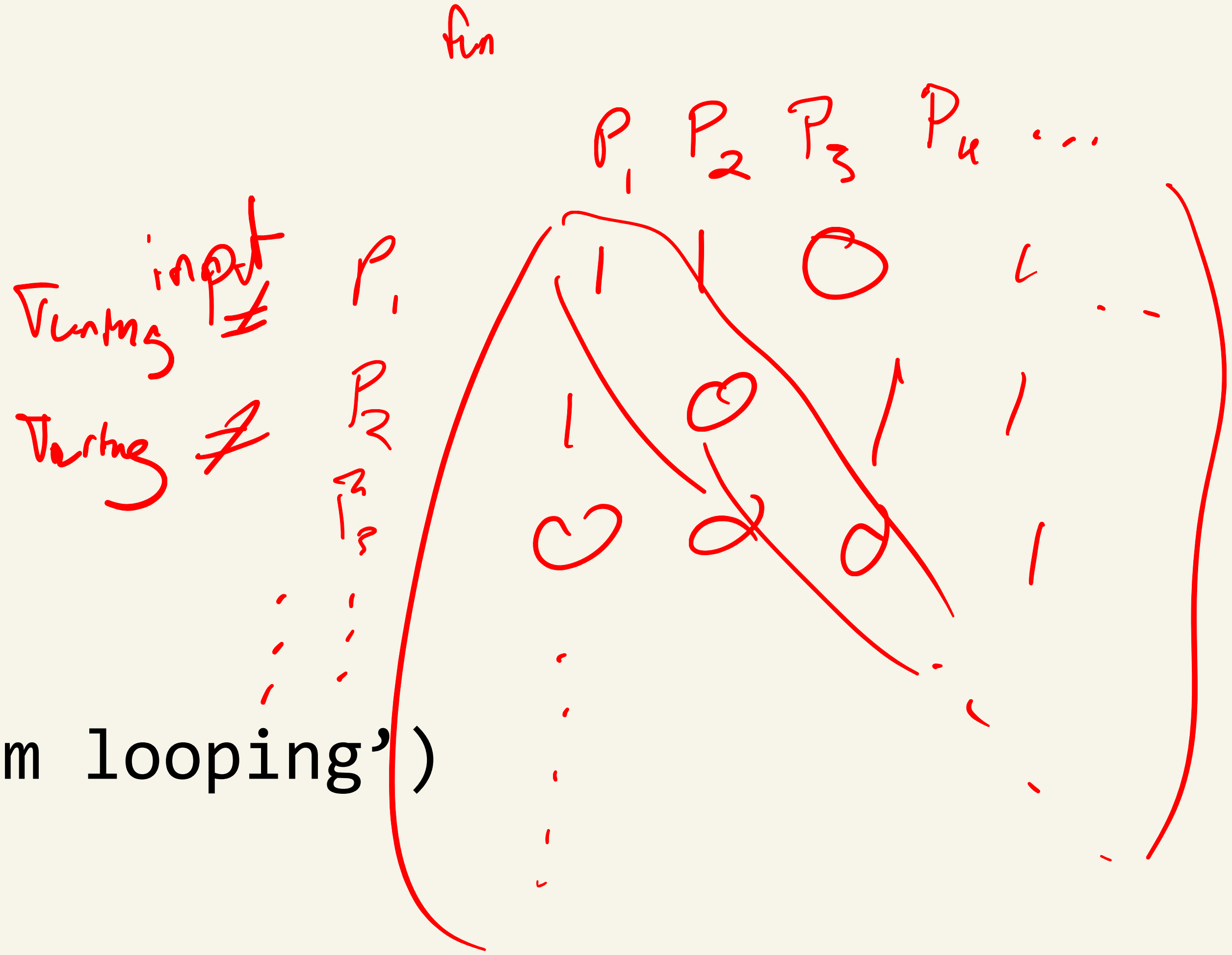
print "Fun a"

The Halting Problem

```
TestHalt(P, x):
    if (program P halts on input x):
        return 1
    else
        return 0
```

The Halting Problem

```
Turing(P):
    if TestHalt(P,P):
        # loop forever
        while 1:
            print('I am looping')
    else:
        # exit
        return 1
```



⇒ Turing P not computable ⇒ TestHalt not computable

"Easy" Halting Problem

```
TestEasyHalt(P):
    if (program P halts on input 0):
        return 1
    else
        return 0
```

$P_1 \quad P_2 \quad P_3$

$\rightarrow \{1, 0, 0, 1 \ldots\}$

"Easy" Halting Problem

```
TestHalt(P, x):
    def P'(y):
        z = P(x)
        return z
    return TestEasyHalt(P')
```

How many computable sets?

Theorem. *The collection of computable sets is countable.*

Corollary. There are uncountably many uncomputable sets.

How many computable things?

The following sets are countable:

1. *The set of computable functions*

2. *The set of computable numbers*

Gödel’s Incompleteness Theorem

Logical paradoxes

“This statement is false”

Gödel’s Incompleteness Theorem

Gödel numbering

Gödel’s Incompleteness Theorem

Gödel numbers of proofs, proof checker

Gödel’s Incompleteness Theorem

Unprovable statement