Markov Chain I

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Example 1a

Toss an unfair coin $\mathbb{P}(\text{Head}) = p$ for N times. What's the fraction of time for observing heads out of all outcomes?

Example 1b

Now we have two unfair coins, each is biased to either head or tail.

Coin 1: $\mathbb{P}(\text{Head}) = p;$ Coin 2: $\mathbb{P}(\text{Head}) = 1 - p.$

If seeing head, then use coin 1 for next toss; if seeing tail, then use coin 2 for the next toss.

Toss N times, what's the fraction of time for observing heads out of all outcomes?

Markov Chain (Discrete Time Finite MC):

State Space:

At each time step n, the state is denoted by $\boldsymbol{X_n}$. The collection of all possible value a state can take is called the state space $S = \{1,2,\ldots,K\}$ for a finite number K.

Transition Probability:

$$P_{ij} = \mathbb{P}(\boldsymbol{X_{n+1}} = j \mid \boldsymbol{X_n} = i) \;\; \forall i,j \in S$$

Markov Property

$$\mathbb{P}(\boldsymbol{X_{n+1}} = j \,|\boldsymbol{X_n} = i, \boldsymbol{X_{n-1}} = i_{n-1}, \dots, \boldsymbol{X_0} = i_0)$$

$$= \mathbb{P}(\boldsymbol{X_{n+1}} = j \,|\boldsymbol{X_n} = i) = P_{ij}$$

Example 1b

Now we have two unfair coins, each is biased to either head or tail.

Coin 1: $\mathbb{P}(\text{Head}) = p$;   Coin 2: $\mathbb{P}(\text{Head}) = 1 - p$.

If seeing head, then use coin 1 for next toss; if seeing tail, then use coin 2 for the next toss.

Toss N times, what's the fraction of time for observing heads out of all outcomes?

Example 2: Alice takes probability class.

Alice is either (1) up-to-date or (2) fall behind

$$P_{11} = 0.8, P_{12} = 0.2, P_{21} = 0.6\ P_{22} = 0.4$$

Probability of being in state j, at time step n

Balance Equation

A distribution π is invariant for the transition probability P if it satisfies the balance equation

$$\pi P = \pi$$

Example 2: Alice takes probability class.

Alice is either (1) up-to-date or (2) fall behind. Find the stationary distribution.

$$P_{11} = 0.8, P_{12} = 0.2, P_{21} = 0.6\ P_{22} = 0.4$$

Properties of Markov Chain

Q1: Does an invariant distribution always exist?

Q2: Is it unique?

Q3: Does π_n approach an invariant distribution?

Properties of Markov Chain

Irreducibility

A Markov chain is irreducible if it can go from any state to any other state, possibly after many steps.

Properties of Markov Chain

<u>(a)periodicity</u>

for an irreducible Markov Chain defined on state space S with transition probability P.

Let

$$d(i) \coloneqq \gcd\{n \geq 1 \mid P^n(i,i) > 0\}$$

Then, $d(i)$ has some value for all i, $d(i) = d$.

If d=1, MC is aperiodic

If d>1, MC is periodic with period d.

Example 3a

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example 3b: Alice studies Markov Chain

Find the invariant distribution of Alice’s study status.

Example 3c: Find the invariant distribution.

A(n) __________________ Markov Chain with ____________________ is aperiodic.

Theorem for Markov Chain

1) If a Markov Chain is finite and irreducible:

2) If this Markov Chain is also aperiodic:

Example 1b. Tossing two unfair coins

One more example

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$