Markov Chain II

Aug 1, 2022

Markov Chain review

- Markov Property
- State Space
- Transition Probability
- Balance Equation
  - Invariant distribution (aka, stationary distribution, steady state distribution
- Irreducible MC has unique Invariant distribution
- Irreducible and aperiodic MC has $\pi_n \rightarrow \pi, n \rightarrow \infty$
- Long- term fraction of time of state i

$$\lim_{N\rightarrow\infty} \{\frac{1}{N} \mathbb{I}\{\boldsymbol{X}_n = i\}\} = \pi(i)$$

Given an irreducible MC, if it contains self loop, then it is aperiodic

- The reserve is not true.
  - counterexample, random walk on a triangle

```
P =

         0    0.5000    0.5000
    0.5000         0    0.5000
    0.5000    0.5000         0
```

```
>> P^2

ans =

    0.5000    0.2500    0.2500
    0.2500    0.5000    0.2500
    0.2500    0.2500    0.5000
```

```
>> P^4

ans =

    0.3750    0.3125    0.3125
    0.3125    0.3750    0.3125
    0.3125    0.3125    0.3750
```

```
>> P^6

ans =

    0.3438    0.3281    0.3281
    0.3281    0.3438    0.3281
    0.3281    0.3281    0.3438
```

```
>> P^3

ans =

    0.2500    0.3750    0.3750
    0.3750    0.2500    0.3750
    0.3750    0.3750    0.2500
```

```
>> P^5

ans =

    0.3125    0.3438    0.3438
    0.3438    0.3125    0.3438
    0.3438    0.3438    0.3125
```

```
>> P^7

ans =

    0.3281    0.3359    0.3359
    0.3359    0.3281    0.3359
    0.3359    0.3359    0.3281
```

```
>> P^100

ans =

    0.3333    0.3333    0.3333
    0.3333    0.3333    0.3333
    0.3333    0.3333    0.3333
```

Given an irreducible MC, if it is aperiodic, then $\pi_n \to \pi, n \to \infty$

- The reserve is not true
  - Counterexample: random walk on a square

```
P =

         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
```

```
>> P^2

ans =

    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
```

```
>> P^3

ans =

         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
```

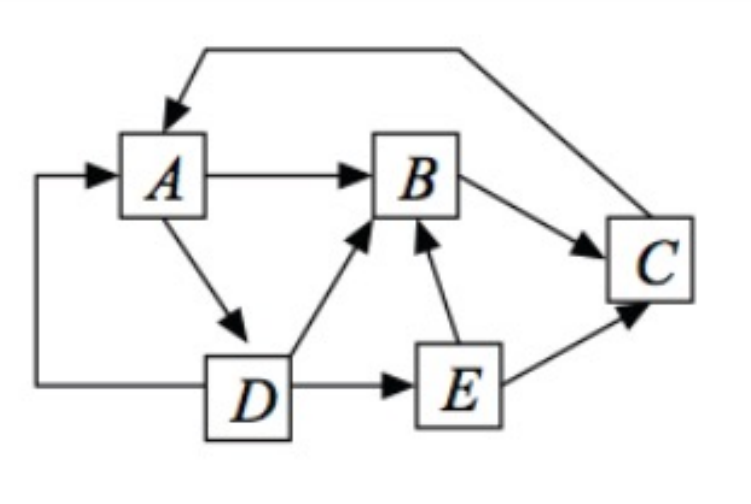
```
>> P^4

ans =

    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
```

```
>> P^5

ans =

         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
```

Example 1

A MC with outgoing arrows are equally likely

1) Is it irreducible?
2) Write transition probability
3) What’s the most frequently visited state?



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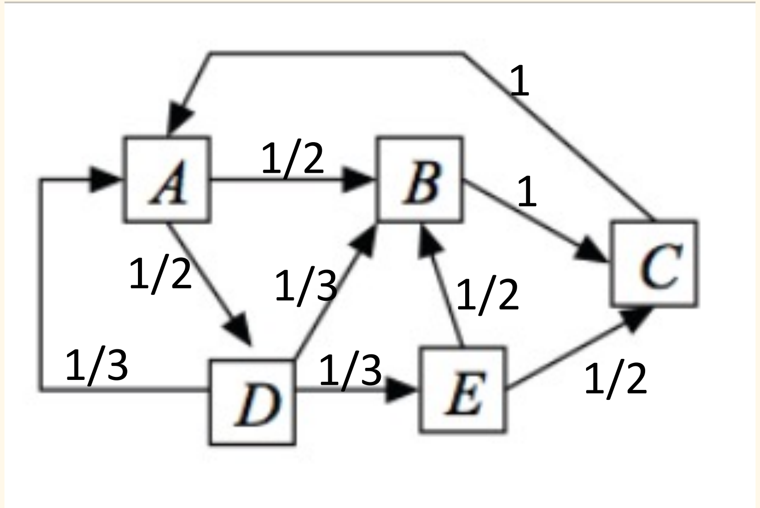
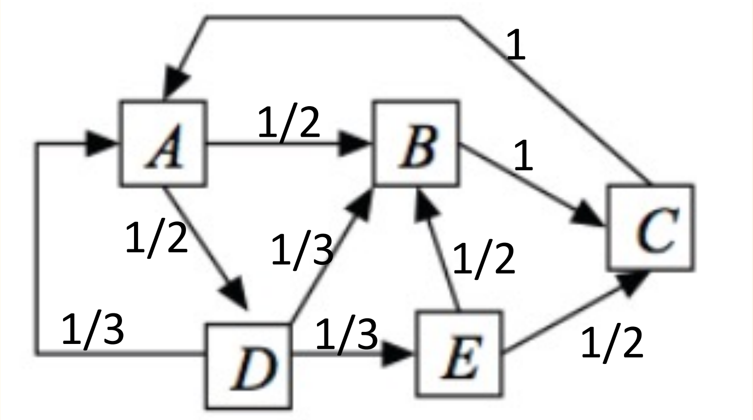


Start at A, how many steps does it take to reach E?

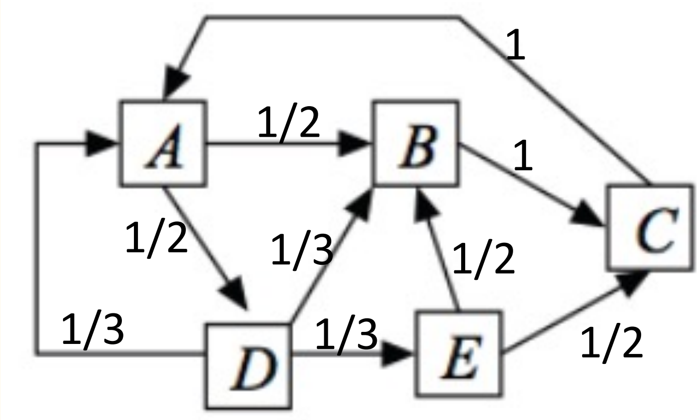


Hitting time of E starting at i is defined as

$$\beta(i) \coloneqq \mathbb{E}(T_E | \boldsymbol{X}_0 = i) \quad \text{for } i = A, B, C, D, E$$

Goal: to calculate $\beta(A) \coloneqq \mathbb{E}(T_E | \boldsymbol{X}_0 = A)$

Hitting time of E starting at i is defined as



$$\beta(i) \coloneqq \mathbb{E}(T_E | \boldsymbol{X}_0 = i) \quad \text{for } i = A, B, C, D, E$$

Goal: to calculate $\beta(A) \coloneqq \mathbb{E}(T_E | \boldsymbol{X}_0 = A)$

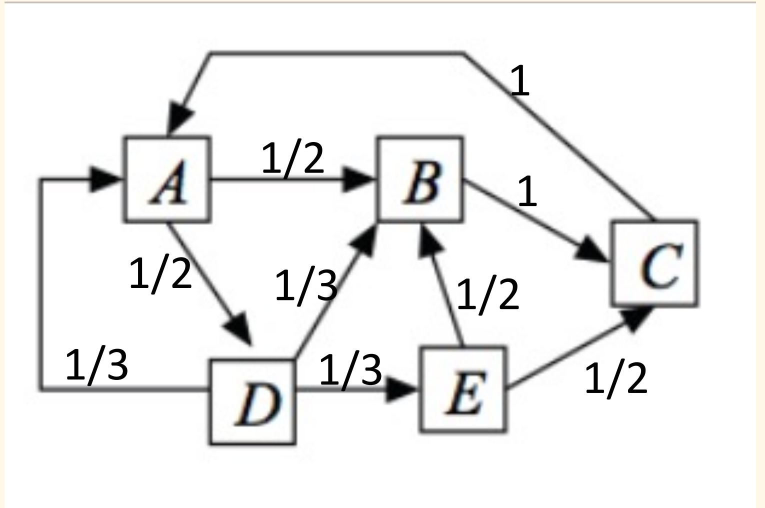
Example 2

Flip a fair coin, how many times on average you need to flip to get two head in a row?

Example 3

Toss a fair 6 face dice, on average, how many times we need to toss until we have the product of two number in a row is 12?

Example 1.



What's the probability that we start at A and we visit C before we visit E

Define:

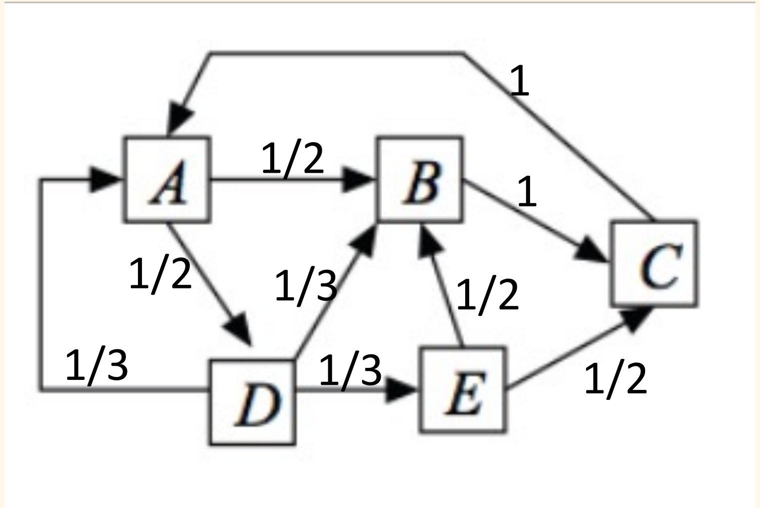
$$\alpha(i) := \mathbb{P}(T_c < T_E | \boldsymbol{X}_0 = i) \quad \text{for } i = A, B, C, D, E$$

Goal: to calculate $\alpha(A) := \mathbb{P}(T_c < T_E | \boldsymbol{X}_0 = A)$

Example 1

$$\alpha(i) \coloneqq \mathbb{P}(T_c < T_E | \boldsymbol{X}_0 = i)$$

$$\text{for } i = A, B, C, D, E$$



Goal: to calculate $\alpha(A) \coloneqq \mathbb{P}(T_c < T_E | \boldsymbol{X}_0 = A)$

General First Step Equation (1)

For a Markov Chain on state space $S = \{1,2, \dots, K\}$ with transition probability P, let T_i be the hitting time of state i.
For a set $A \subset S$ of states, let $T_A = \min\{n \geq 0 | \boldsymbol{X}_n \in A\}$ be the hitting time of the set A.

1) We consider the mean value of T_A

General First Step Equation (2)

For a Markov Chain on state space $S = \{1,2, \dots, K\}$ with transition probability P, let T_i be the hitting time of state i.
For a set $A \subset S$ of states, let $T_A = \min \{n \geq 0 | \boldsymbol{X}_n \in A\}$ be the hitting time of the set A.

2) We consider the probability of hitting set A before B

Not required *

General First Step Equation (3)

3) We consider collecting an amount of $h(i)$ every time visiting state i before visiting state A

$$Y = \sum_{n=0}^{T_A} h(\boldsymbol{X}_n)$$

Not required *

General First Step Equation (4)

4) We consider a discount factor β for moving one step

$$Z = \sum_{n=0}^{T_A} \beta^n h(\boldsymbol{X}_n)$$