1 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \land (Q \lor P) \equiv P \land Q$

(b) $(P \lor Q) \land R \equiv (P \land R) \lor (Q \land R)$

(c) $(P \land Q) \lor R \equiv (P \lor R) \land (Q \lor R)$

Solution:

(a) Not equivalent.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \land (Q \lor P)$</th>
<th>$P \land Q$</th>
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</thead>
<tbody>
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(b) Equivalent.

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<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$(P \land Q) \land R$</th>
<th>$(P \land R) \lor (Q \land R)$</th>
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(c) Equivalent.
2 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) \((\forall x \in \mathbb{Z}) \ (x \in \mathbb{Q})\)

(e) \((\forall x \in \mathbb{Z}) \ (((2 \mid x) \lor (3 \mid x)) \implies (6 \mid x))\)

(f) \((\forall x \in \mathbb{N}) \ ((x > 7) \implies ((\exists a, b \in \mathbb{N}) \ (a + b = x)))\)

Solution:

(a) \((\exists x \in \mathbb{R}) \ (x \notin \mathbb{Q})\), or equivalently \((\exists x \in \mathbb{R}) \ \neg (x \in \mathbb{Q})\). This is true, and we can use \(\pi\) as an example to prove it.

(b) \((\forall x \in \mathbb{Z}) \ (((x \in \mathbb{N}) \lor (x < 0)) \land \neg ((x \in \mathbb{N}) \land (x < 0)))\). This is true, since we define the naturals to contain all integers which are not negative.

(c) \((\forall x \in \mathbb{N}) \ ((6 \mid x) \implies ((2 \mid x) \lor (3 \mid x)))\). This is true, since any number divisible by 6 can be written as \(6k = (2 \cdot 3)k = 2(3k)\), meaning it must also be divisible by 2.

(d) All integers are rational numbers. This is true, since any integer number \(n\) can be written as \(n/1\).

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false—2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.
If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since we can take \( a = x \) and \( b = 0 \).

(Aside: this is a reference to the very weak Goldback Conjecture (https://xkcd.com/1310/).

### 3 Converse and Contrapositive

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication \( P \implies Q \) is \( \neg P \implies \neg Q \).)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

**Solution:**

The notation \( a \mid b \) ("\( a \) divides \( b \)) denotes that \( b \) is divisible by \( a \).

(a) \( (\forall x \in \mathbb{N}) (4 \mid x \implies 2 \mid x) \). This statement is true. We know that if \( x \) is divisible by 4, we can write \( x \) as \( 4k \) for some integer \( k \). But \( 4k = (2 \cdot 2)k = 2(2k) \), where \( 2k \) is also an integer. Thus, \( x \) must also be divisible by 2, since it can be written as 2 times an integer.

(b) The inverse is that if a natural number is not divisible by 4, it is not divisible by 2: \( (\forall x \in \mathbb{N}) (4 \nmid x \implies 2 \nmid x) \). This is false, since 2 is not divisible by 4, but is divisible by 2.

(c) The converse is that any natural number that is divisible by 2 is also divisible by 4: \( (\forall x \in \mathbb{N}) (2 \mid x \implies 4 \mid x) \). Again, this is false, since 2 is divisible by 2 but not by 4.

(d) The contrapositive is that any natural number that is not divisible by 2 is not divisible by 4: \( (\forall x \in \mathbb{N}) (2 \nmid x \implies 4 \nmid x) \). To show that this is true, first consider that saying that \( x \) is not divisible by 2 is equivalent to saying that \( x/2 \) is not an integer. And if we divide a non-integer by an integer, we get back another non-integer—so \( x/2 \) must also not be an integer. But that is exactly the same as saying that \( x \) is not divisible by 4.

Note that the inverse and the converse will always be contrapositives of each other, and so will always be logically equivalent.
4 Equivalences with Quantifiers

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

| (a) | \( \forall x \left( (\exists y \ Q(x,y)) \Rightarrow P(x) \right) \) | \( \forall x \exists y \ (Q(x,y) \Rightarrow P(x)) \) |
| (b) | \( \neg \exists x \forall y \ (P(x,y) \Rightarrow \neg Q(x,y)) \) | \( \forall x \left( \left( \exists y P(x,y) \right) \land \left( \exists y Q(x,y) \right) \right) \) |
| (c) | \( \forall x \exists y \ (P(x) \Rightarrow Q(x,y)) \) | \( \forall x \ (P(x) \Rightarrow (\exists y Q(x,y))) \) |

Solution:

(a) Not equivalent.

**Justification:** We can rewrite the left side as \( \forall x \left( (\neg (\exists y Q(x,y))) \lor P(x) \right) \) and the right side as \( \forall x \exists y \left( \neg Q(x,y) \lor P(x) \right) \). Applying the negation on the left side of the equivalence \( (\neg (\exists y Q(x,y))) \) changes the \( \exists y \) to \( \forall y \), and the two sides are clearly not the same. Another approach to the problem is to consider by linguistic example. Let \( x \) and \( y \) span the universe of all people, and let \( Q(x,y) \) mean “\( x \) is \( y \)’s offspring”, and let \( P(x) \) mean “\( x \) likes tofu”. The right side claims that, for all Persons \( x \), there exists some Person \( y \) such that either Person \( x \) is not \( y \)’s offspring or that Person \( x \) likes tofu. The left side claims that, for all Persons \( x \), if there exists a parent of Person \( x \), then Person \( x \) likes tofu. Obviously, these are not the same.

(b) Not equivalent.

**Justification:** Using De Morgan’s Law to distribute the negation on the left side yields

\[ \forall x \exists y \left( P(x,y) \land \neg Q(x,y) \right). \]

But \( \exists \) does not distribute over \( \land \). There could exist different values of \( y \) such that \( P(x,y) \) and \( Q(x,y) \) for a given \( x \), but not necessarily the same value.

(c) Equivalent.

**Justification:** We can rewrite the left side as \( \forall x \exists y \left( \neg P(x) \lor Q(x,y) \right) \) and the right side as \( \forall x \left( \left( \neg P(x) \right) \lor \left( \exists y Q(x,y) \right) \right) \). Clearly, the two sides are the same if \( \neg P(x) \) is true. If \( \neg P(x) \) is false, then the two sides are still the same, because \( \forall x \exists y \left( \text{False} \lor Q(x,y) \right) \equiv \forall x \left( \text{False} \lor \left( \exists y Q(x,y) \right) \right) \).