Propositional Logic Intro

Note 2

Proposition: A statement with a truth value; it is either true or false.

Propositions can be combined to form more complicated expressions, using the following operations:

Operators		Quantifiers		Implication operations	
$\wedge$	and	$\forall$	for all	Implication	$P \implies Q$
$\vee$	or	$\exists$	there exists	Inverse	$\neg P \implies \neg Q$
$\neg$	not			Converse	$Q \implies P$
$\implies$	implies			Contrapositive	$\neg Q \implies \neg P$
$\equiv$	equivalent to				

Further, for an implication $P \implies Q$ where P is the *hypothesis* and Q is the *conclusion*, it is useful to know that $P \implies Q \equiv \neg P \vee Q$. Additionally, observe that any implication is logically equivalent to its contrapositive.

DeMorgan's Laws: The following identities can be helpful when simplifying expressions and distributing negations.

- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(\forall x\, P(x)) \equiv \exists x(\neg P(x))$
- $\neg(\exists x\, P(x)) \equiv \forall x(\neg P(x))$

1 Propositional Practice

Note 2

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

Recall that $\mathbb{R}$ is the set of reals, $\mathbb{Q}$ is the set of rationals, $\mathbb{Z}$ is the set of integers, and $\mathbb{N}$ is the set of natural numbers. The notation "$a \mid b$", read as "a divides b", means that a is a divisor of b.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $\neg(\forall x \in \mathbb{Q})(x \in \mathbb{Z})$

(e) $(\forall x \in \mathbb{Z})(((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

Solution:

(a) $(\exists x \in \mathbb{R})\ (x \notin \mathbb{Q})$, or equivalently $(\exists x \in \mathbb{R})\ \neg(x \in \mathbb{Q})$. This is true, and we can use π as an example to prove it.

(b) $(\forall x \in \mathbb{Z})\ (((x \in \mathbb{N}) \vee (x < 0)) \wedge \neg((x \in \mathbb{N}) \wedge (x < 0)))$. This is true, since we define the naturals to contain all integers which are not negative.

(c) $(\forall x \in \mathbb{N})\ ((6 \mid x) \implies ((2 \mid x) \vee (3 \mid x)))$. This is true, since any number divisible by 6 can be written as $6k = (2 \cdot 3)k = 2(3k)$, meaning it must also be divisible by 2.

(d) It is not the case that all rational numbers are integers. This is true, since we have rational numbers like $\frac{1}{2}$ that are not integers. The statement can be rewritten as $(\exists x \in \mathbb{Q})\neg(x \in \mathbb{Z})$, which can be read as "there exists a rational number that is not in the set of integers".

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false: 2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.

(f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since if $x = a + b$ we can take $a = x$ and $b = 0$.

(Aside: this is a reference to the very weak Goldback Conjecture (https://xkcd.com/1310/).)

2 Implication

Note 2

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(c) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

Solution:

(a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

(b) False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(c) True. The antecedent says that there is an x, say x', where for every y, $P(x,y)$ is true. If the antecedent is true, then for every y one can choose $x = x'$, which will make the consequent true.

Proofs Intro

Note 1
Note 3

In this course, we'll be working a lot with proofs: a *proof* is a sequence of logical deductions which establishes that a particular statement is true. Here are four proof strategies that we typically use to prove that a statement is true.

1. **Direct Proof**: To prove $P \implies Q$, assume P, $[\cdots]$, therefore Q.
2. **Proof by Contraposition**: To prove $P \implies Q$, it is equivalent to prove $\neg Q \implies \neg P$. As such, assume $\neg Q$, $[\cdots]$, therefore $\neg P$.

   A proof by contraposition relies on the fact that the contrapositive is equivalent to the original implication: $P \implies Q \equiv \neg Q \implies \neg P$.
3. **Proof by Contradiction**: To prove P, assume $\neg P$, $[\cdots]$, therefore R, $[\cdots]$, therefore $\neg R$. This is a contradiction, since R and $\neg R$ cannot both be true, so P must be true.

   In a proof by contradiction, the propositions R and $\neg R$ often arise naturally as you construct the proof; in the majority of cases, you do not decide on R beforehand, and it can often be hard to determine what R should be.
4. **Proof by Cases**: Divide the problem into parts, and tackle each separately. Oftentimes, each case will involve one of the other proof types.

As you get familiar with these proof techniques, it can often be unclear which strategy to take: this is expected! It's natural to start out through trial and error: attempt a proof strategy, and if it doesn't work out, try another. There is no magical rule to tell you how to prove a statement: as you write more and more proofs, you'll likely find some patterns that indicate that a particular proof strategy may be most suitable for the claim. The best way to get better at writing proofs is to practice!

3 Numbers of Friends

Note 3

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.

(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), is friends with a person who has 0 friends (i.e., friends with no one). This is a contradiction since friendship is mutual.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0, 1, \ldots, n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.

4 Pebbles

Note 3

Suppose you have a rectangular 2D array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones.

Prove that there must exist an all-red column.

Solution: We give a proof by contraposition; the contrapositive is "if there does not exist an all-red column, then there is always a way of choosing one pebble from each column such that there does not exist a red pebble among the chosen ones".

Suppose there does not exist an all-red column. This means that we can always find a blue pebble in each column. Therefore, if we take one blue pebble from each column, we have a way of choosing one pebble from each column without any red pebbles. This is the negation of the original hypothesis, so we are done.

We can also approach the problem through contradiction; the logic stays almost exactly the same, and we start with the negation of the conclusion: that there does not exist an all-red column. The same reasoning above allows us to conclude that there will always exist a way of choosing one pebble from each column such that all pebbles are blue (i.e. no pebbles are red).

5 Minimum of Two Reals

Note 3

Prove that $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x+y-|x-y|)/2$.

Solution: We will use a proof by cases. Recall that the definition of the absolute value function for real number z is

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case 1: $x < y$. This means $|x-y| = y-x$. Substituting this into the formula on the right hand side, we get

$$\frac{x+y-y+x}{2} = x = \min(x,y).$$

Case 2: $x \geq y$. This means $|x-y| = x-y$. Substituting this into the formula on the right hand side, we get

$$\frac{x+y-x+y}{2} = y = \min(x,y).$$