Propositional Logic Intro

Note 2

Proposition: A statement with a truth value; it is either true or false.

Propositions can be combined to form more complicated expressions, using the following operations:

Operators		Quantifiers		Implication operations	
$\wedge$	and	$\forall$	for all	Implication	$P \implies Q$
$\vee$	or	$\exists$	there exists	Inverse	$\neg P \implies \neg Q$
$\neg$	not			Converse	$Q \implies P$
$\implies$	implies			Contrapositive	$\neg Q \implies \neg P$
$\equiv$	equivalent to				

Further, for an implication $P \implies Q$ where P is the *hypothesis* and Q is the *conclusion*, it is useful to know that $P \implies Q \equiv \neg P \vee Q$. Additionally, observe that any implication is logically equivalent to its contrapositive.

DeMorgan's Laws: The following identities can be helpful when simplifying expressions and distributing negations.

- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(\forall x\, P(x)) \equiv \exists x(\neg P(x))$
- $\neg(\exists x\, P(x)) \equiv \forall x(\neg P(x))$

1 Propositional Practice

Note 2

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

Recall that $\mathbb{R}$ is the set of reals, $\mathbb{Q}$ is the set of rationals, $\mathbb{Z}$ is the set of integers, and $\mathbb{N}$ is the set of natural numbers. The notation "$a \mid b$", read as "a divides b", means that a is a divisor of b.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $\neg(\forall x \in \mathbb{Q})(x \in \mathbb{Z})$

(e) $(\forall x \in \mathbb{Z})(((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

2 Implication

Note 2

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(c) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

Proofs Intro

Note 1
Note 3

In this course, we'll be working a lot with proofs: a *proof* is a sequence of logical deductions which establishes that a particular statement is true. Here are four proof strategies that we typically use to prove that a statement is true.

1. **Direct Proof**: To prove $P \implies Q$, assume P, $[\cdots]$, therefore Q.
2. **Proof by Contraposition**: To prove $P \implies Q$, it is equivalent to prove $\neg Q \implies \neg P$. As such, assume $\neg Q$, $[\cdots]$, therefore $\neg P$.

   A proof by contraposition relies on the fact that the contrapositive is equivalent to the original implication: $P \implies Q \equiv \neg Q \implies \neg P$.
3. **Proof by Contradiction**: To prove P, assume $\neg P$, $[\cdots]$, therefore R, $[\cdots]$, therefore $\neg R$. This is a contradiction, since R and $\neg R$ cannot both be true, so P must be true.

   In a proof by contradiction, the propositions R and $\neg R$ often arise naturally as you construct the proof; in the majority of cases, you do not decide on R beforehand, and it can often be hard to determine what R should be.
4. **Proof by Cases**: Divide the problem into parts, and tackle each separately. Oftentimes, each case will involve one of the other proof types.

As you get familiar with these proof techniques, it can often be unclear which strategy to take: this is expected! It's natural to start out through trial and error: attempt a proof strategy, and if it doesn't work out, try another. There is no magical rule to tell you how to prove a statement: as you write more and more proofs, you'll likely find some patterns that indicate that a particular proof strategy may be most suitable for the claim. The best way to get better at writing proofs is to practice!

3 Numbers of Friends

Note 3

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.

(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

4 Pebbles

Note 3

Suppose you have a rectangular 2D array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones.

Prove that there must exist an all-red column.

5 Minimum of Two Reals

Note 3

Prove that $\forall x, y \in \mathbb{R}$, $\min(x, y) = (x + y - |x - y|)/2$.