1 Contraposition

Prove the statement "if $a + b < c + d$, then $a < c$ or $b < d$".

**Solution:**

The implication we’re trying to prove is $(a + b < c + d) \implies ((a < c) \lor (b < d))$, so the contrapositive is $((a \geq c) \land (b \geq d)) \implies (a + b \geq c + d)$. The proof of this is quite straightforward: since we have both that $a \geq c$ and that $b \geq d$, we can just add these two inequalities together, giving us $a + b \geq c + d$, which is exactly what we wanted.

2 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.

(Hint: The Pigeonhole Principle states that if $n$ items are placed in $m$ containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

**Solution:**

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n - 1$, we conclude that for every $i \in \{0, 1, \ldots, n - 1\}$, there is exactly one person who has exactly $i$ friends at the party. In particular, there is one person who has $n - 1$ friends (i.e., friends with everyone), is friends with a person who has 0 friends (i.e., friends with no one). This is a contradiction since friendship is mutual.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to $n$ possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0, 1, \ldots, n - 1$. The objects assigned to these containers are the people at the party. However, containers $0, n - 1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning $n$ people to at most $n - 1$ containers, and by the pigeonhole principle, at least one of the $n - 1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.
3 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

**Solution:** We give a proof by contraposition. Suppose there does not exist an all-red column. This means that, in each column, we can find a blue pebble. Therefore, if we take one blue pebble from each column, we have a way of choosing one pebble from each column without any red pebbles. This is the negation of the original hypothesis, so we are done.

4 Preserving Set Operations

For a function \( f \), define the image of a set \( X \) to be the set \( f(X) = \{ y \mid y = f(x) \text{ for some } x \in X \} \).

Define the inverse image or preimage of a set \( Y \) to be the set \( f^{-1}(Y) = \{ x \mid f(x) \in Y \} \). Prove the following statements, in which \( A \) and \( B \) are sets.

Recall: For sets \( X \) and \( Y \), \( X = Y \) if and only if \( X \subseteq Y \) and \( Y \subseteq X \). To prove that \( X \subseteq Y \), it is sufficient to show that \((\forall x) ((x \in X) \implies (x \in Y))\).

(a) \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \).

(b) \( f(A \cup B) = f(A) \cup f(B) \).

**Solution:**

In order to prove equality \( A = B \), we need to prove that \( A \) is a subset of \( B \), \( A \subseteq B \) and that \( B \) is a subset of \( A \), \( B \subseteq A \). To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose \( x \in f^{-1}(A \cup B) \) which means that \( f(x) \in A \cup B \). Then either \( f(x) \in A \), in which case \( x \in f^{-1}(A) \), or \( f(x) \in B \), in which case \( x \in f^{-1}(B) \), so in either case we have \( x \in f^{-1}(A) \cup f^{-1}(B) \). This proves that \( f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B) \).

Now, suppose that \( x \in f^{-1}(A) \cup f^{-1}(B) \). Suppose, without loss of generality, that \( x \in f^{-1}(A) \). Then \( f(x) \in A \), so \( f(x) \in A \cup B \), so \( x \in f^{-1}(A \cup B) \). The argument for \( x \in f^{-1}(B) \) is the same. Hence, \( f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B) \).

(b) Suppose that \( x \in A \cup B \). Then either \( x \in A \), in which case \( f(x) \in f(A) \), or \( x \in B \), in which case \( f(x) \in f(B) \). In either case, \( f(x) \in f(A) \cup f(B) \), so \( f(A \cup B) \subseteq f(A) \cup f(B) \).

Now, suppose that \( y \in f(A) \cup f(B) \). Then either \( y \in f(A) \) or \( y \in f(B) \). In the first case, there is an element \( x \in A \) with \( f(x) = y \); in the second case, there is an element \( x \in B \) with \( f(x) = y \). In either case, there is an element \( x \in A \cup B \) with \( f(x) = y \), which means that \( y \in f(A \cup B) \). So \( f(A) \cup f(B) \subseteq f(A \cup B) \).
The purpose of this problem is to gain familiarity to naming things precisely. In particular, we named an element in the LHS (or the pre-image of the LHS) and then argued about whether that element or its image was in the right hand side. By explicitly naming an element generically where it could be *any* element in the set, we could argue about its membership in a set and or its image or preimage. With these different concepts floating around it is helpful to be clear in the argument.