CS 70 Discrete Mathematics and Probability Theory
Fall 2021 DIS 1A

1 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \ \equiv \ P \wedge Q$

(b) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$

(c) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$

Solution:

(a) Not equivalent.

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

(b) Equivalent.

P	Q	R	$(P \vee Q) \wedge R$	$(P \wedge R) \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(c) Equivalent.

P	Q	R	$(P \wedge Q) \vee R$	$(P \vee R) \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	F

2 Converse and Contrapositive

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

Solution:

(a) $(\forall x \in \mathbb{N})\ (4 \mid x \implies 2 \mid x)$. This statement is true. We know that if x is divisible by 4, we can write x as $4k$ for some integer k. But $4k = (2 \cdot 2)k = 2(2k)$, where $2k$ is also an integer. Thus, x must also be divisible by 2, since it can be written as 2 times an integer.

(b) The inverse is that if a natural number is not divisible by 4, it is not divisible by 2: $(\forall x \in \mathbb{N})\ (4 \nmid x \implies 2 \nmid x)$. This is false, since 2 is not divisible by 4, but is divisible by 2.

(c) The converse is that any natural number that is divisible by 2 is also divisible by 4: $(\forall x \in \mathbb{N})\ (2 \mid x \implies 4 \mid x)$. Again, this is false, since 2 is divisible by 2 but not by 4.

(d) The contrapositive is that any natural number that is not divisible by 2 is not divisible by 4: $(\forall x \in \mathbb{N})\ (2 \nmid x \implies 4 \nmid x)$. To show that this is true, first consider that saying that x is not divisible by 2 is equivalent to saying that $x/2$ is not an integer. And if we divide a non-integer by an integer, we get back another non-integer–so $(x/2)/2 = x/4$ must also not be an integer. But that is exactly the same as saying that x is not divisible by 4.

Note that the inverse and the converse will always be contrapositives of each other, and so will always be logically equivalent.

3 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f(A \cup B) = f(A) \cup f(B)$.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose $x \in f^{-1}(A \cup B)$ which means that $f(x) \in A \cup B$. Then either $f(x) \in A$, in which case $x \in f^{-1}(A)$, or $f(x) \in B$, in which case $x \in f^{-1}(B)$, so in either case we have $x \in f^{-1}(A) \cup f^{-1}(B)$. This proves that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Suppose, without loss of generality, that $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(x) \in A \cup B$, so $x \in f^{-1}(A \cup B)$. The argument for $x \in f^{-1}(B)$ is the same. Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

(b) Suppose that $x \in A \cup B$. Then either $x \in A$, in which case $f(x) \in f(A)$, or $x \in B$, in which case $f(x) \in f(B)$. In either case, $f(x) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now, suppose that $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. In the first case, there is an element $x \in A$ with $f(x) = y$; in the second case, there is an element $x \in B$ with $f(x) = y$. In either case, there is an element $x \in A \cup B$ with $f(x) = y$, which means that $y \in f(A \cup B)$. So $f(A) \cup f(B) \subseteq f(A \cup B)$.

The purpose of this problem is to gain familiarity to naming things precisely. In particular, we named an element in the LHS (or the pre-image of the LHS) and then argued about whether that element or its image was in the right hand side. By explicitly naming an element generically where it could be *any element in the set, we could argue about its membership in a set and or its image or preimage. With these different concepts floating around it is helpful to be clear in the argument.*

4 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.

(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0,1,\dots,n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0,1,...,n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.