1 Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1 + x)^n \geq 1 + nx$.

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that $a_n \leq 3^{2^n}$ for every positive integer $n$.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{2^n}$? Attempt an induction proof with this hypothesis to show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{2^n-1}$ using induction.

(c) Why does the hypothesis in part (b) imply the overall claim?
3 Binary Numbers

Prove that every positive integer \( n \) can be written in binary. In other words, prove that we can write

\[ n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0, \]

where \( k \in \mathbb{N} \) and \( c_i \in \{0, 1\} \) for all \( i \leq k \).

4 Fibonacci for Home

Recall, the Fibonacci numbers, defined recursively as

\[ F_1 = 1, \ F_2 = 1, \ \text{and} \ F_n = F_{n-2} + F_{n-1}. \]

Prove that every third Fibonacci number is even. For example, \( F_3 = 2 \) is even and \( F_6 = 8 \) is even.